

CYCLES IN THE DE RHAM COHOMOLOGY OF ABELIAN VARIETIES OVER NUMBER FIELDS

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ABSTRACT. In his 1982 paper, Ogus defined a class of cycles in the de Rham cohomology of smooth proper varieties over number fields. This notion is a crystalline analogue of ℓ -adic Tate cycles. In the case of abelian varieties, this class includes all the Hodge cycles by the work of Deligne, Ogus, and Blasius. Ogus predicted that such cycles coincide with Hodge cycles for abelian varieties. In this paper, we confirm Ogus' prediction for some families of abelian varieties, under the assumption that these cycles lie in Betti cohomology with real coefficients. These families include abelian varieties that have both prime dimension and nontrivial endomorphism ring. The proof is based on a crystalline analogue of Faltings' isogeny theorem due to Bost and the known cases of the Mumford–Tate conjecture.

1. INTRODUCTION

Given a polarized abelian variety A over a number field K , the Mumford–Tate conjecture for A concerns the ℓ -adic monodromy group G_ℓ , which is defined to be the Zariski closure of the image of the Galois representation of $\mathrm{Gal}(\bar{K}/K)$ on the ℓ -adic Tate module of A . The conjecture predicts that the connected component G_ℓ° of G_ℓ coincides with the base change $G_{\mathrm{MT}} \otimes \mathbb{Q}_\ell$ of Mumford–Tate group of A . In terms of cycles, the conjecture asserts that ℓ -adic Tate cycles are \mathbb{Q}_ℓ -linear combinations of Hodge cycles. Here elements in the tensor algebra of ℓ -adic étale cohomology are called *ℓ -adic Tate cycles* if they are fixed by $\mathrm{Gal}(\bar{L}/L)$ after suitable Tate twist for some finite extension L of K .

As a crystalline analogue, Ogus defined the notion of absolute Tate cycles for any smooth projective variety X over a number field K and predicted that, via the de Rham–Betti comparison, absolute Tate cycles coincide with absolute Hodge cycles ([Ogu82, Hope 4.11.3]). For any finite extension L of K , an element in the tensor algebra of $\bigoplus_{i=0}^{2\dim X} H_{\mathrm{dR}}^i(X/K) \otimes L$ is called an *absolute Tate cycle* (*loc. cit.* Def. 4.1) if it is fixed by all but finitely many crystalline Frobenii φ_v . When v is unramified and X has good reduction at v , the Frobenius φ_v can be viewed as an action on $H_{\mathrm{dR}}^i(X/K) \otimes K_v$ via the canonical isomorphism between the de Rham and the crystalline cohomologies. He proved that all the Hodge cycles are absolute Tate for abelian varieties and verified his prediction when X is the product of abelian varieties with complex multiplication, Fermat hypersurfaces, and projective spaces (*loc. cit.* Thm. 4.16).

It is natural to take the archimedean places into account: complex conjugation on the Betti cohomology can be viewed as the analogue of Frobenius action on the crystalline cohomology. We define *de Rham–Tate cycles* (2.1.1) to be absolute Tate cycles that satisfy in addition that, for any embedding $\sigma : K \rightarrow \mathbb{C}$, they lie in the

image of the tensor algebra of $\bigoplus_{i=0}^{2\dim X} H_B^i(X_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$ under the comparison isomorphism $H_B^i(X_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H_{\text{dR}}^i(X/K) \otimes_K \mathbb{C}$.

Our first result is the following:

Theorem 1.1 (Theorem 3.2.4). *If A is a polarized abelian variety over \mathbb{Q} and its ℓ -adic algebraic monodromy group G_ℓ is connected, then the Mumford–Tate conjecture for A implies that the de Rham–Tate cycles coincide with the Hodge cycles.*

The Mumford–Tate conjecture for abelian varieties is known in many cases by the work of many people including Serre, Chi, Ribet, Tankeev, Pink, Banaszak, Gajda, Krasoń, Vasiu. We refer the reader to [Pin98], [Vas08], [BGK06], [BGK10] and their references. Some fundamental ingredients in the proofs are Faltings’ isogeny theorem, p -adic Hodge theory, and Serre’s theory of Frobenius tori. When the abelian variety A over K satisfies $\text{End}_{\bar{K}}(A) = \mathbb{Z}$, Pink constructs a \mathbb{Q} -model of G_ℓ° which is independent of ℓ and “looks like” G_{MT} in the following sense. The group G_{MT} (resp. the \mathbb{Q} -model of G_ℓ°) with its tautological faithful absolutely irreducible representation $H_B^1(A, \mathbb{Q})$ (resp. $H_{\text{ét}}^1(A_{\bar{K}}, \mathbb{Q}_\ell)$) is an (absolutely) irreducible *strong Mumford–Tate pair* over \mathbb{Q} : the group is reductive and generated over \mathbb{Q} by the image of a cocharacter of weights $(0, 1)$. Here weights mean the weights of the cocharacter composed with the faithful representation.

In the crystalline setting, we define the de Rham–Tate group G_{dR} of a polarized abelian variety A over K to be the algebraic subgroup over K of $\text{GL}(H_{\text{dR}}^1(A/K))$ stabilizing all of the de Rham–Tate cycles. This group is reductive by our assumption that de Rham–Tate cycles are fixed by complex conjugation. We have the following result similar to those by Pink:

Theorem 1.2. *Assume that A is a polarized abelian variety over \mathbb{Q} and the ℓ -adic algebraic monodromy group of A is connected. If $\text{End}_{\mathbb{Q}}(A) = \mathbb{Z}$, then the neutral connected component of G_{dR} with its tautological representation is an irreducible strong Mumford–Tate pair over \mathbb{Q} .*

Theorem 3.2.6 is a more general version of this theorem. Based on the work of Serre, Pink gave a classification of irreducible Mumford–Tate pairs. See [Pin98, Prop. 4.4, 4.5, and Table 4.6]. Although the conclusion of Theorem 1.2 is a consequence of Theorem 1.1, this theorem, along with Pink’s classification, shows unconditionally that G_{dR} is of a very restricted form. Beyond the $K = \mathbb{Q}$ case, we have:

Theorem 1.3. *Let A be a polarized abelian variety over some number field isogenous to $\prod_{i=1}^n A_i^{n_i}$, where A_i is absolutely simple and A_i is not isogenous to A_j over any number field for $i \neq j$. Assume that each A_i is one of the following cases:*

- (1) A_i is an elliptic curve or has complex multiplication.
- (2) A_i does not have complex multiplication and the dimension of A_i is a prime number and $\text{End}_{\bar{K}}(A_i)$ is not \mathbb{Z} .
- (3) The polarized abelian variety A_i of dimension g with $\text{End}_{\bar{K}}(A_i) = \mathbb{Z}$ is defined over a finite Galois extension K over \mathbb{Q} such that $[K : \mathbb{Q}]$ is prime to $g!$ and $2g$ is not of form a^{2b+1} or $\binom{4b+2}{2b+1}$ where $a, b \in \mathbb{N} \setminus \{0\}$.

and that if there is an A_i of case (2) and of type IV in Albert’s classification, then all the other A_j are not of type IV. Then the de Rham–Tate cycles of A coincide with its Hodge cycles.

Case (1) was known before. We now explain the main difficulty when K is not equal to \mathbb{Q} . For simplicity, we focus on the case that $\text{End}_{\bar{K}}(A) = \mathbb{Z}$. Pink's classification applies to connected reductive groups with an absolutely irreducible representation. In analogy with Faltings' isogeny theorem, Bost proved that all the absolute Tate cycles lying in $\text{End}(H_{\text{dR}}^1(A/K) \otimes L)$ are algebraic ([Bos06, Thm. 6.4]). We can deduce from this the irreducibility of $H_{\text{dR}}^1(A/K)$ as a G_{dR} -representation. However, G_{dR} is a priori not connected. In the ℓ -adic setting, Serre, using the Chebotarev density theorem, showed that G_{ℓ} will be connected after passing to a finite extension. There seems to be no easily available analogue argument for G_{dR} .

When $K = \mathbb{Q}$, the absolute Frobenii coincide with the relative ones. Thus the connectedness of G_{ℓ} implies that G_{dR} is *almost connected*: $\varphi_p \in G_{\text{dR}}^{\circ}(\mathbb{Q}_p)$ for all p inside a set of natural density 1. The almost connectedness implies that $H_{\text{dR}}^1(A/\mathbb{Q}) \otimes \bar{\mathbb{Q}}$ is even absolutely irreducible as a G_{dR} -representation. This is a consequence of a strengthening of Bost's result, which we prove based on the work of Bost, Gasbarri and Herblot: for any abelian variety A over K , if $s \in \text{End}(H_{\text{dR}}^1(A/K))$ satisfies that $\varphi_v(s) = s$ for all v over p in a set of rational primes of natural density one, then s is algebraic.

For a general K , the group G_{ℓ} only contains information about the relative Frobenii $\varphi_v^{m_v}$, where $m_v = [K_v : \mathbb{Q}_p]$. Without additional inputs, it is only reasonable to expect that the connectedness of G_{ℓ} would imply that a variant of G_{dR} defined using $\varphi_v^{m_v}$ is almost connected. In analogy with Bost's theorem, one may expect:

Conjecture 1.4. *If $s \in \text{End}(H_{\text{dR}}^1(A, L))$ is fixed by all but finitely many relative Frobenii, then s is an L -linear combination of algebraic cycles.*

One can prove a generalization of Theorem 1.2 for an arbitrary K using this conjecture. However, it seems difficult to prove the conjecture in general.

To illustrate the idea of the proof of Theorem 1.3, we focus on the case when $A_{\bar{K}}$ is simple. Case (1) was known before our work. For the rest, the main task is to show that the centralizer of G_{dR}° in $\text{End}(H_{\text{dR}}^1(A/K))$ coincides with that of G_{dR} . In the case (2), since the Mumford–Tate group is not too large, we use Bost's theorem to show that otherwise G_{dR}° must be a torus. Then we deduce that A must have complex multiplication using a theorem of Noot ([Noo96, Thm. 2.8]) on formal deformation spaces at a point of ordinary reduction. To exploit the strengthening of Bost's result to tackle case (3), we need to understand φ_v for all v lying over $p \in M$, where M is some set of rational primes of density 1. While Serre's theorem on the ranks of Frobenius tori only provides information about completely split primes, we prove a refinement when $G_{\ell} = \text{GSp}_{2g}$ that takes into account the other primes. Our result asserts that the Frobenius tori are of maximal rank for all v lying over $p \in M$. The rest of the argument is similar to that of the case (2).

Organization of the paper. In section 2, given an abelian variety A over a number field K , we construct the category of motives generated by A with morphisms being the de Rham–Tate cycles following Deligne's construction with absolute Hodge cycles. We prove that this category is a semisimple Tannakian category whose fundamental group is $G_{\text{dR}} \subset \text{GSp}(H_{\text{dR}}^1(A/K))$. From this we obtain our reformulation of Bost's result: the centralizer of G_{dR} in $\text{End}(H_{\text{dR}}^1(A/K))$ coincides with that of G_{MT} . Then we prove the strengthening of Bost's result. In the end, we discuss a variant of de Rham–Tate cycles defined using relative Frobenii.

In section 3, we recall the Mumford–Tate conjecture and the theory of Frobenius tori initiated by Serre. A result of Katz–Messing [KM74] enables us to view the Frobenius tori as a subgroup of both G_{dR} and G_ℓ and serves as a bridge between results for G_ℓ and those for G_{dR} . We first prove a refinement of results of Serre and Chi on the rank of Frobenius tori. Then we prove Theorem 1.1 and 1.2. In the end, we recall the result of Noot and discuss its application.

In section 4, we prove Theorem 1.3. In the first half, we study the irreducible G_{dR}° -subrepresentations of $H_{\text{dR}}^1(A/K) \otimes \bar{K}$. This part is valid for a large class of abelian variety without assuming the Mumford–Tate conjecture. A main input is the action of Frobenius tori using the result of Pink that G_ℓ with its tautological representation is a weak Mumford–Tate pair over \mathbb{Q}_ℓ . In the second half, we complete the proof of Theorem 1.3.

In section 5, we further generalize the result of Bost using the idea of Gasbarri and Herblot and discuss its consequence on the de Rham–Tate cycles for abelian surfaces over quadratic fields, the first case when Theorem 1.3 does not apply.

Notation and Convention. Let A be a polarized abelian variety of dimension g defined over a number field K . For any vector space V , let V^\vee be its dual and we denote $V^{\otimes m} \otimes (V^\vee)^{\otimes n}$ by $V^{m,n}$. For any algebraic group G or vector space V over K , we denote by G_R or V_R the base change to any K -algebra R .

We assume that L is a finite extension of K if there is no specific indication and denote by $H_{\text{dR}}^i(A, L)$ the de Rham cohomology $H_{\text{dR}}^i(A_L/L) = \mathbb{H}^i(A_L, \Omega_{A_L/L}^\bullet) = H_{\text{dR}}^i(A/K) \otimes_K L$. Let v be a finite place of L and k_v be the residue field. If A_L has good reduction at v , we use φ_v to denote the crystalline Frobenius action on $H_{\text{dR}}^1(A, L_v)^{m,n}$ via the canonical isomorphism to the crystalline cohomology $(H_{\text{cris}}^1(A_{k_v}/W(k_v)) \otimes L_v)^{m,n}$. For any archimedean place σ corresponding to an embedding $\sigma: L \rightarrow \mathbb{C}$, let φ_σ be the map on de Rham cohomology induced by the complex conjugation on the Betti cohomology.

The polarization of A gives a natural identification between $H_{\text{dR}}^1(A, K)$ and $(H_{\text{dR}}^1(A, K))^\vee(-1)$, where (-1) denotes the Tate twist, preserving filtration and Frobenius actions.

We use A^\vee to denote the dual abelian variety of A and $E(A)$ to denote the universal vector extension of A .

For convenience, a reductive algebraic group in our paper could be nonconnected. For any algebraic group G , we use G° and $Z(G)$ to denote the neutral connected component and the center of G . When G is reductive, by the rank of G we mean the rank of any maximal torus of G .

For any field F , we use \bar{F} to denote a chosen algebraic closure of F . For any finite dimensional vector space V over F and any subset S of V , we use $\text{Span}_F(S)$ to denote the smallest sub F -vector space of V containing S .

Sometimes, we use the abbreviation dRT to stand for de Rham–Tate.

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2. DE RHAM–TATE CYCLES AND A RESULT OF BOST

2.1. De Rham–Tate cycles.

Definition 2.1.1. An element $s \in (H_{\text{dR}}^1(A, L))^{m,n}$ is called a *de Rham–Tate cycle* of the abelian variety A (over L) if there exists a finite set Σ of finite places of L such that for all places $v \notin \Sigma$, $\varphi_v(s) = s$.

Remark 2.1.2.

- (1) Similar to [Ogu82, Cor. 4.8.1, 4.8.3], we see that $s \in (H_{\text{dR}}^1(A, K))^{m,n}$ is de Rham–Tate if and only if so is its base change in $(H_{\text{dR}}^1(A, L))^{m,n}$ and the set of de Rham–Tate cycles over L is stable under the natural Galois action of $\text{Gal}(L/K)$ (on the coefficient of de Rham cohomology groups).
- (2) Due to [Ogu82, Cor. 4.8.2], although one could define de Rham–Tate cycles over arbitrary field L containing K , we only need to consider cycles over number fields since any de Rham–Tate cycle must be defined over $\bar{\mathbb{Q}}$ and hence over some number field.

We have the following important fact, which we sketch a proof for completeness.

Lemma 2.1.3 ([Ogu82, Prop. 4.15]). *If $s \in H_{\text{dR}}^1(A, L)^{m,n}$ is fixed by infinitely many φ_v (in particular, when s is de Rham–Tate), then s lies in $\text{Fil}^0 H_{\text{dR}}^1(A, L)^{m,n}$. Moreover, if such s lies in $\text{Fil}^1 H_{\text{dR}}^1(A, L)^{m,n}$, then $s = 0$.*

Proof. By [Maz73, Thm. 7.6] and the extension of Mazur’s theorem to $H_{\text{dR}}^1(A, L)^{m,n}$ in the proof by Ogus, we have that for all but finitely many v , the mod \mathfrak{p} filtration $\text{Fil}^j((H_{\text{cris}}^1(A_{k_v}/W(k_v)) \otimes k_v)^{m,n})$ is the same as the following set¹ modulo \mathfrak{p}

$$\{\xi \in (H_{\text{cris}}^1(A_{k_v}/W(k_v)))^{m,n} \text{ such that } \varphi_v(\xi) \in p^j(H_{\text{cris}}^1(A_{k_v}/W(k_v)))^{m,n}\}$$

. Then for the infinitely many v such that $\varphi_v(s) = s$, we have

$$s \in \text{Fil}^0((H_{\text{cris}}^1(A_{k_v}/W(k_v)) \otimes k_v)^{m,n})$$

and if $s \in \text{Fil}^1 H_{\text{dR}}^1(A, L)^{m,n}$, then s is 0 modulo \mathfrak{p} . Since the Hodge filtration over L is compatible with the Hodge filtration over k_v , we obtain the desired assertions. \square

The main conjecture that we are aiming for is the following:

Conjecture 2.1.4. *The set of de Rham–Tate cycles of an abelian variety A defined over K coincides with the set of Hodge cycles via the isomorphism between Betti and de Rham cohomologies.*

Remark 2.1.5.

- (1) Our conjecture is weaker than the conjectures of Ogus [Ogu82, Problem 2.4, Hope 4.11.3]. Therefore, 2.1.4 was known when A has complex multiplication ([Ogu82, Thm. 4.16]). It was also known when A is an elliptic curve. See [And04, 7.4.3.1] for an explanation using Serre–Tate theory.
- (2) This conjecture reduces to the case when A is principally polarized. To see this, we notice that after passing to some finite extension of K , the abelian variety A is isogenous to a principally polarized one. Moreover this conjecture is insensitive to base change and the conjectures for two isogenous abelian varieties are equivalent.

¹The polarization of A gives a natural $W(k_v)$ -module structure of the dual of $H_{\text{cris}}^1(A_{k_v}/W(k_v))$. See also the proof of Ogus for more details.

Theorem 2.1.6 ([Del82, Theorem 2.11], [Ogu82, Theorem 4.14], [Bla94]). *For any abelian variety, every Hodge cycle is de Rham–Tate.*

Therefore, the open part of 2.1.4 is whether all of the de Rham–Tate cycles are Hodge cycles.

2.2. The de Rham–Tate group. We fix an isomorphism of K -vector spaces $H_{\text{dR}}^1(A, K)$ and K^{2g} . Then the algebraic group $\text{GL}_{2g, K}$ acts on $H_{\text{dR}}^1(A, K)$ and hence on $H_{\text{dR}}^1(A, L)^{m, n}$.

Definition 2.2.1. We define G_{dR} to be the algebraic subgroup of $\text{GL}_{2g, \bar{K}}$ such that for any \bar{K} -algebra R , its R -points $G_{\text{dR}}(R)$ is the subgroup of $\text{GL}_{2g}(R)$ which fixes all de Rham–Tate cycles. We call G_{dR} the *de Rham–Tate group* of the abelian variety A .

Remark 2.2.2.

- (1) The group G_{dR} is algebraic since $G_{\text{dR}} = \bigcap_{s_\alpha} G_\alpha$, where s_α runs through all de Rham–Tate cycles and G_α is the algebraic subgroup of $\text{GL}_{2g, \bar{K}}$ which fixes s_α .
- (2) The de Rham–Tate group G_{dR} is naturally defined over K by 2.1.2 (1). From now on, we use G_{dR} to denote the K -algebraic group.

Lemma 2.2.3. *There exists a smallest number field K^{dR} containing K such that all of the de Rham–Tate cycles are defined over K^{dR} . Moreover, the algebraic group G_{dR} can be defined as the stabilizer of a finite set of de Rham–Tate cycles $\{s_\alpha\}$ and K^{dR} is the smallest number field such that all these s_α are defined. Furthermore, K^{dR} is Galois over K .*

Proof. Let K^{dR} be the smallest number field over which all s_α in the finite set are defined. We need to show that if $t \in (H_{\text{dR}}^1(A, \bar{K}))^{m, n}$ is de Rham–Tate, then t is defined over K^{dR} . Let L be a number field such that t is defined and we may assume L is Galois over K^{dR} . Let W be the smallest sub vector space of $(H_{\text{dR}}^1(A, K^{\text{dR}}))^{m, n}$ such that $t \in W \otimes L$. Let Γ be the Galois group $\text{Gal}(L/K^{\text{dR}})$. Then $W \otimes L$ is spanned by γt for $\gamma \in \Gamma$. By 2.1.2 (1), these γt are de Rham–Tate, and hence $W \otimes L$ is spanned by de Rham–Tate cycles. Then by definition, $G_{\text{dR}}(L)$ acts on $W \otimes L$ trivially and hence so does $G_{\text{dR}}(K^{\text{dR}})$ on W . On the other hand, since $\{s_\alpha\} \cup \{t\}$ is a finite set, for all but finitely many finite places v , we have $\varphi_v(s_\alpha) = s_\alpha$ and $\varphi_v(t) = t$. Let p be the residue characteristic of v and let m_v be $[K_v^{\text{dR}} : \mathbb{Q}_p]$. The K_v^{dR} -linear action $\varphi_v^{m_v}$ lies in $G_{\text{dR}}(K_v^{\text{dR}})$ since it fixes all s_α and hence acts on $W \otimes K_v^{\text{dR}}$ trivially. By definition, $\varphi_v(t) = t$ and hence t is stable by $\varphi_v^{m_v}$. Therefore, t is defined over K^{dR} by the Chebotarev density theorem (applying to the coefficients of t expressed in terms of a K^{dR} -base of W). The last assertion of the lemma comes from 2.1.2. \square

2.2.4. In order to reformulate 2.1.4 in terms of algebraic groups following the idea of Deligne, we briefly recall the definition and basic properties of the Mumford–Tate group G_{MT} , which is closely related to the Hodge cycles. See [Del82, Sec. 3] for details. When we discuss G_{MT} and Hodge cycles, we always fix an embedding $\sigma : K \rightarrow \mathbb{C}$. We denote $H_B^1(A(\mathbb{C}), \mathbb{Q})$ by V_B , which has a natural polarized Hodge structure of type $((1, 0), (0, 1))$. Let $\mu : \mathbb{G}_{m, \mathbb{C}} \rightarrow \text{GL}(V_{B, \mathbb{C}})$ be the *Hodge cocharacter*, through which $z \in \mathbb{C}^\times$ acts by multiplication with z on $V_{B, \mathbb{C}}^{1, 0}$ and trivially

on $V_{B,\mathbb{C}}^{0,1}$. The *Mumford–Tate group* G_{MT} of the abelian variety A is the smallest algebraic subgroup defined over \mathbb{Q} of $\text{GL}(V_B)$ such that its base change to \mathbb{C} containing the image of μ . The Mumford–Tate group is the algebraic subgroup of $\text{GL}(V_B)$ which fixes all Hodge cycles.² Since all Hodge cycles are absolute Hodge cycles in the abelian variety case ([Del82, Thm. 2.11]), the algebraic group G_{MT} is independent of the choice of σ .

Corollary 2.2.5. *Via the de Rham–Betti comparison, we have $G_{\text{dR},\mathbb{C}} \subset G_{\text{MT},\mathbb{C}}$ and the Hodge cocharacter μ factors through $G_{\text{dR},\mathbb{C}}$.*

Proof. It follows from Theorem 2.1.6 and Lemma 2.1.3. \square

2.2.6. The Mumford–Tate group G_{MT} is reductive ([Del82, Prop. 3.6]) and the fixed part of G_{MT} in $V_B^{m,n}$ is the set of Hodge cycles. 2.1.4 is equivalent to the following conjecture, which we will mainly focus on from now on.

Conjecture 2.2.7. *Via the de Rham–Betti comparison, we have $G_{\text{dR},\mathbb{C}} \cong G_{\text{MT},\mathbb{C}}$.*

Proof of equivalence. 2.1.4 implies this conjecture. Conversely, by 2.2.6, the isomorphism of these two groups implies that every \mathbb{C} -linear combination of de Rham–Tate cycles maps to a \mathbb{C} -linear combination of Hodge cycles via the de Rham–Hodge comparison. Then we conclude by Theorem 2.1.6 and Prop. 4.9 in [Ogu82]. \square

Remark 2.2.8. This conjecture implies that G_{dR} would be connected and reductive. We will show that G_{dR} is reductive using the same idea of the proof of [Del82, Prop. 3.6]. However, there seems no direct way to show that G_{dR} is connected without proving the above conjecture first.

Lemma 2.2.9. *The de Rham–Tate group G_{dR} is reductive.*

Proof. Fix an embedding $\sigma : K \rightarrow \mathbb{C}$. By Corollary 2.2.5, we view $G_{\text{dR},\mathbb{C}}$ as a subgroup of $G_{\text{MT},\mathbb{C}} \subset \text{GL}(V_{B,\mathbb{C}})$. Since all de Rham–Tate cycles are fixed by φ_σ , the subgroup $G_{\text{dR},\mathbb{C}}$ descends to an \mathbb{R} -subgroup. Therefore, both μ and its complex conjugate $\bar{\mu}$ factor through $G_{\text{dR},\mathbb{C}}$ by Corollary 2.2.5 and so does $h = \mu \cdot \bar{\mu}$. Let ψ be the polarization on V_B and $G_{\text{dR},\mathbb{C}}^1$ be the subgroup of $G_{\text{dR},\mathbb{C}}$ acting trivially on the Tate twist. Then ψ is invariant under $G_{\text{dR},\mathbb{C}}^1$. Let C be $h(i) \in G_{\text{dR},\mathbb{C}}^1$ and let $\phi(x, y)$ be $\psi(x, Cy)$. Then the positive definite form ϕ on $V_{B,\mathbb{R}}$ is invariant under $\text{ad}C(G_{\text{dR},\mathbb{C}}^1)(\mathbb{R})$ ³. Therefore, $G_{\text{dR},\mathbb{C}}^1$ has a compact real form $\text{ad}C(G_{\text{dR},\mathbb{C}}^1)(\mathbb{R})$ and is reductive. Then $G_{\text{dR}} = G_{\text{dR}}^1 \cdot Z(G_{\text{dR}})$ is reductive. \square

2.3. The centralizer of the de Rham–Tate group. The following proposition, whose proof uses the construction of a Tannakian category of de Rham–Tate cycles, provides a description of the centralizer of G_{dR} in $\text{End}(H_{\text{dR}}^1(A, K))$. We will use this proposition to reformulate a result of Bost. Moreover, at the end of this subsection, we sketch a proof a strengthening of the result of Bost that will be used to describe the centralizer of G_{dR}° . The proof uses some arguments in section 5, where we provide a complete proof of an even stronger version (Corollary 5.2.2).

²Here we consider Hodge cycles as elements in $V_B^{m,m-2i}(i) \subset V_B^{m',n'}$ for some choice of m', n' as Tate twists is a direct summand of the tensor algebra of V_B .

³One can check by definition that the \mathbb{C} -sub group $\text{ad}C(G_{\text{dR},\mathbb{C}}^1)$ is an \mathbb{R} -subgroup of $G_{\text{MT},\mathbb{C}}$.

Proposition 2.3.1. *Let s be an element in $(H_{\text{dR}}^1(A, L))^{m,n}$ for some number field L containing K^{dR} . The de Rham–Tate group G_{dR} fixes s if and only if s is a L -linear combination of de Rham–Tate cycles.*

2.3.2. We now construct the category $\mathcal{M}_{\text{dRT},L}$ of motives of de Rham–Tate cycles of the abelian variety A , where L is a field algebraic over K . We follow the idea of the construction of the motive of absolute Hodge cycles in [DM82, Sec. 6]. Let $\langle A \rangle^\otimes$ be the set of varieties generated by A under finite product and disjoint union, and let $H_{\text{dR}}(X)$ be the direct sum of $H_{\text{dR}}^i(X, L)$ for all i .

The objects in the category $\mathcal{M}_{\text{dRT},L}$ are

$$M = (X, n, pr), \text{ where } n \in \mathbb{Z}, X \in \langle A \rangle^\otimes, pr \in \text{End}(H_{\text{dR}}(X, L)) \text{ idempotent dRT.}$$

The set of morphisms $\text{Hom}(M_1, M_2)$ is defined to be

$$\{f : H_{\text{dR}}(X_1)(n_1) \rightarrow H_{\text{dR}}(X_2)(n_2) \text{ de Rham–Tate such that } f \circ pr_1 = pr_2 \circ f\} / \sim,$$

where \sim is defined by modulo $\{f : f \circ pr_1 = 0 = pr_2 \circ f\}$.

[Ogu82, Prop. 4.9] shows that that $\mathcal{M}_{\text{dRT},L}$ is \mathbb{Q} -linear with $\text{End}(\mathbb{I}) = \mathbb{Q}$, where $\mathbb{I} = (pt, 0, id)$ and that $\text{Hom}(M_1, M_2)$ is a finite dimensional \mathbb{Q} -vector space. Moreover, by the above construction, the category $\mathcal{M}_{\text{dRT},L}$ is a pseudo-abelian rigid tensor category (see also [And04, 4.1.3, 4.1.4]). Since the de Rham–Tate cycles lie in the image of the Betti cohomology with real coefficients under the Betti–de Rham comparison, $pr(H_{\text{dR}}(X, \mathbb{C}))$ has a real Hodge structure. By [DM82, Prop. 6.2] and the fact that absolute Hodge cycles are de Rham–Tate cycles, $pr(H_{\text{dR}}(X, \mathbb{C}))$ is polarized. Hence $\text{End}(M)$ is semi-simple by [DM82, Prop. 4.5, Prop. 6.3]. Therefore, we use [Jan92, Lem. 2] to conclude that $\mathcal{M}_{\text{dRT},L}$ is a rigid abelian tensor category. By [Del90, Thm. 1.12], this is a Tannakian category with a fiber functor $\omega_L : M \mapsto pr(H_{\text{dR}}(X, L))$ over L . Let G_{dR}^L be the Tannakian fundamental group $\underline{\text{Aut}}^\otimes(\omega_L)$. Since $\mathcal{M}_{\text{dRT},L}$ is semi-simple, G_{dR}^L is a reductive algebraic group over L .

2.3.3. We now describe the relation between de Rham–Tate groups of cycles over different fields. One can define de Rham–Tate cycles on zero dimensional varieties as in 2.1.1 and define the motive $\mathcal{M}_{\text{dRT},L}^0$ as above. This category is the category of Artin motives and we denote by $\Gamma(L)$ its Tannakian fundamental group, which is an L -form of the Galois group $\text{Gal}(\bar{L}/L)$. A modification of the proof of [DM82, Prop. 6.23] shows that the following sequence is exact:

$$1 \rightarrow G_{\text{dR}}^{\bar{L}} \rightarrow G_{\text{dR}}^L \rightarrow \overline{\Gamma(L)} \rightarrow 1,$$

where $\overline{\Gamma(L)}$ is a quotient of $\Gamma(L)$. More precisely, $\overline{\Gamma(L)}$ is the Tannakian fundamental group of $\mathcal{M}_{\text{dRT},L} \cap \mathcal{M}_{\text{dRT},L}^0$.

The category that we will mainly focus on is $\mathcal{M}_{\text{dRT},\bar{K}}$, which is equivalent to $\mathcal{M}_{\text{dRT},K^{\text{dR}}}$ by Lemma 2.2.3, and we will denote them by \mathcal{M}_{dRT} .

Proof of Proposition 2.3.1. Let $\{s_\alpha\}$ be the set of de Rham–Tate cycles. We view $G_{\text{dR}}^{K^{\text{dR}}}$ as a subgroup of $\text{GL}(H_{\text{dR}}^1(A, K))$ (a priori, $G_{\text{dR}}^{K^{\text{dR}}}$ is only defined over K^{dR} , but it descends to K by 2.1.2(1)). Since \mathcal{M}_{dRT} is Tannakian, we have an equivalence of categories

$$\mathcal{M}_{\text{dRT}} \otimes L \cong \text{Rep}_L(G_{\text{dR}}^{K^{\text{dR}}}).$$

Hence that s is an L -linear combination of s_α is equivalent to that s is fixed by $G_{\text{dR}}^{K^{\text{dR}}}$ and it remains to prove that $G_{\text{dR}}^{K^{\text{dR}}} = G_{\text{dR}}$. Since G_{dR} is defined to be the

stabilizer of all s_α , the above equivalence of categories shows that $G_{\text{dR}}^{K^{\text{dR}}} \subset G_{\text{dR}}$. Since $G_{\text{dR}}^{K^{\text{dR}}}$ is reductive, then by a theorem of Chevalley, $G_{\text{dR}}^{K^{\text{dR}}}$ is the stabilizer of a line in some direct sum of $(H_{\text{dR}}^1(A, L))^{m,n}$. By the definition of $G_{\text{dR}}^{K^{\text{dR}}}$, this line must be a L -linear combinations of some s_α and hence $G_{\text{dR}}^{K^{\text{dR}}} = G_{\text{dR}}$ because G_{dR} stabilizes all linear combinations of s_α . \square

Remark 2.3.4.

- (1) A by-product of the proof is that G_{dR} is reductive. This argument is essentially the same as the one we gave before since the key input for both arguments is that de Rham–Tate cycles are fixed by φ_σ .
- (2) There is a variant of Proposition 2.3.1 when L is not assumed to contain K^{dR} . More precisely, G_{dR}^L is the largest subgroup of $\text{GL}(H_{\text{dR}}^1(A, L))$ that stabilizes all de Rham–Tate cycles over L and $s \in (H_{\text{dR}}^1(A, L))^{m,n}$ is an L -linear combination of de Rham–Tate cycles over L if and only if s is fixed by the action of G_{dR}^L .

Motivated by [Her, Def. 3.5], we have

Definition 2.3.5. An element s of $(H_{\text{dR}}^1(A, L))^{m,n}(i)$ is called a β -de Rham–Tate cycles if

$$\beta \leq \liminf_{x \rightarrow \infty} \left(\sum_{v, p_v \leq x, \varphi_v(s)=s} \frac{[L_v : \mathbb{Q}_{p_v}] \log p_v}{p_v - 1} \right) \left([L : \mathbb{Q}] \sum_{p \leq x} \frac{\log p}{p - 1} \right)^{-1},$$

where v (resp. p) runs over finite places of L (resp. \mathbb{Q}) and p_v is the rational prime under v .

Remark 2.3.6.

- (1) Absolute Tate cycles and hence de Rham–Tate cycles are 1-de Rham–Tate cycles by definition.
- (2) Assume that M is a set of rational primes with natural density β and for any $v|p \in M$, $\varphi_v(s) = s$. Then s is a β -de Rham–Tate cycle by [Her, Lem. 3.7].

Theorem 2.3.7. *The set of 1-de Rham–Tate cycles in $\text{End}(H_{\text{dR}}^1(A, L))$ is the image of $\text{End}_L(A) \otimes \mathbb{Q}$. In particular, the centralizer of G_{dR} in $\text{End}(H_{\text{dR}}^1(A, L))$ is $\text{End}_L(A) \otimes L$.*

Proof. The second assertion follows from the first one by Proposition 2.3.1 and the above remark. The first statement restricted to absolute Tate cycles is a direct consequence of [Bos01, Thm. 2.3] and we refer the reader to [And04, 7.4.3] for a proof. See also [Bos06, Thm. 6.4]. Notice that their argument is valid for 1-de Rham–Tate cycles if one can generalize [Bos01, Thm. 2.3] to the following

Claim. Given a commutative algebraic group G over L and an L -sub vector space W of $\text{Lie } G$. Assume that there exists a set M of finite places of L such that:

- (1) for any $v \in M$ over rational prime p , W modulo v is closed under p -th power map,

$$(2) \liminf_{x \rightarrow \infty} \left(\sum_{v, p_v \leq x, v \in M} \frac{[L_v : \mathbb{Q}_{p_v}] \log p_v}{p_v - 1} \right) \left([L : \mathbb{Q}] \sum_{p \leq x} \frac{\log p}{p - 1} \right)^{-1} = 1.$$

Then W is the Lie algebra of some algebraic subgroup of G .

This claim follows from Theorem 5.1.5, which is a refinement of a theorem of Gasbarri by incorporating ideas of Herblot. We refer the reader to section 5 for the proof of Theorem 5.1.5 and the definition of the terminologies used below. Now we explain how to obtain the claim. The idea is due to Bost. We apply Theorem 5.1.5 to the formal leaf \hat{V} passing through identity of the involutive subbundle of the tangent bundle of G generated by W via translation. We take the uniformization map to be the exponential map $W(\mathbb{C}) \rightarrow \text{Lie } G(\mathbb{C}) \rightarrow G$. It is a standard fact that the order ρ of this uniformization map is finite.⁴ On the other hand, the assumptions on W are equivalent to $\alpha = 0$. There would be a contradiction with Theorem 5.1.5 if \hat{V} is not algebraic. \square

2.4. Relative de Rham–Tate cycles. Let L be a finite extension over K . Let v be a finite place of L with residue characteristic p and define $m_v = [L_v : \mathbb{Q}_p]$. We have an L_v -linear endomorphism, the relative Frobenius $\varphi_v^{m_v}$, of $H_{\text{dR}}^1(A, L_v)$ and hence of $(H_{\text{dR}}^1(A, L_v))^{m,n}$.

Definition 2.4.1. An element $t \in (H_{\text{dR}}^1(A, L))^{m,n}$ is called a *relative de Rham–Tate cycle* (over L) of A if there exists a finite set Σ of finite places of L such that for every finite place $v \notin \Sigma$ and every archimedean place σ , one has $\varphi_v^{m_v}(t) = t$ and $\varphi_v^{m_v}(\varphi_\sigma(t)) = \varphi_\sigma t$.

Remark 2.4.2. By definition, any L -linear combination of de Rham–Tate cycles over L is relatively de Rham–Tate. Moreover, for any $\gamma \in \text{Gal}(L/K)$, the cycle $\gamma(t)$ is relatively de Rham–Tate if (and only if) t is so.

In analogy with the definition of de Rham–Tate groups, we have:

Definition 2.4.3. We define G^L to be the algebraic subgroup of $\text{GL}_{2g,L}$ such that any L -algebra R , its R -points $G^L(R)$ is the subgroup of $\text{GL}_{2g}(R)$ which fixes all relative de Rham–Tate cycles t_α over L . We call G^L the *relative de Rham–Tate group* of the abelian variety A over L .

Lemma 2.4.4. *Similar to the corresponding statements for the de Rham–Tate group, we have:*

- (1) *The relative de Rham–Tate group G^L is contained in G_{dR}^L .*
- (2) *Every relative de Rham–Tate cycle lies in $\text{Fil}^0((H_{\text{dR}}^1(A, L))^{m,n})$ and hence the Hodge cocharacter factors through G^L .*
- (3) *The group G^L is the smallest reductive algebraic subgroup of $\text{GL}_{2g,L}$ such that its L_v -points contains $\varphi_v^{m_v}$ for all but finitely many finite places v and that it is stable under φ_σ for all archimedean places σ .*
- (4) *Any element in $(H_{\text{dR}}^1(A, L))^{m,n}$ is relatively de Rham–Tate if and only if it is fixed by the action of G^L .*

Proof. Part (1) follows from 2.4.2. Lemma 2.1.3 implies part (2). To show that G^L is reductive, we notice that φ_σ fixes the set of relative de Rham–Tate cycles and hence the embedding $G_{\mathbb{C}}^L \subset G_{\text{MT},\mathbb{C}}$ is induced from an embedding of \mathbb{R} -groups. Then, combined with part (2), we see that $\mu(i) \cdot \bar{\mu}(i) \in G^L(\mathbb{C})$. Now as in the proof of Lemma 2.2.9, the adjoint action of $\mu(i) \cdot \bar{\mu}(i)$ defines a real form of G^L which is

⁴In [BW07, p. 112], they summarized some results of Faltings and Wütholz that may enable us to show ρ is finite by standard complex analytic arguments.

compact modulo center and hence G^L is reductive. The rest of (3) is direct and it implies (4). \square

3. FROBENIUS TORI AND THE MUMFORD–TATE CONJECTURE

In this section, we use Σ to denote a finite set of finite places of K^{dR} including all ramified places such that for $v \notin \Sigma$, the abelian variety $A_{K^{\text{dR}}}$ has good reduction at v and the Frobenius φ_v stabilizes all of the de Rham–Tate cycles. For any finite extension L of K in question, we still use Σ to denote the finite set of finite places $f^{-1}g(\Sigma)$, where $f : \text{Spec } L \rightarrow \text{Spec } K$ and $g : \text{Spec } K^{\text{dR}} \rightarrow \text{Spec } K$. When we discuss the relative de Rham–Tate cycles, we also enlarge Σ so that the relative Frobenius $\varphi_v^{m_v}$ stabilizes all relative de Rham–Tate cycles over L .

3.1. Frobenius Tori. The following definition is due to Serre. See also [Chi92, Sec. 3] and [Pin98, Sec. 3] for details.

Definition 3.1.1. Let T_v be the Zariski closure of the subgroup of $G_{L_v}^L$ (hence also in G_{dR, L_v}^L) generated by the L_v -linear map $\varphi_v^{m_v} \in G^L(L_v)$. Since $\varphi_v^{m_v}$ is semisimple, the group T_v° is a torus and is called the *Frobenius torus* associated to v .

Remark 3.1.2. The torus T_v° and its rank are independent of the choice of L .

3.1.3. For every prime ℓ , we have the ℓ -adic Galois representation

$$\rho_\ell : \text{Gal}(\bar{K}/K) \rightarrow \text{GL}(H_{\text{ét}}^1(A_{\bar{K}}, \mathbb{Q}_\ell)),$$

and we denote by $G_\ell(A)$ the Zariski closure of the image of $\text{Gal}(\bar{K}/K)$ and call G_ℓ the ℓ -adic monodromy group. If it is clear which variety is concerned, we may just use G_ℓ to denote this group. Serre proved that there exists a smallest finite Galois extension $K^{\text{ét}}$ of K such that for any ℓ , the Zariski closure of the image of $\text{Gal}(\bar{K}^{\text{ét}}/K^{\text{ét}})$ is connected ([Ser81, Sec. 5, p. 15]).

Remark 3.1.4. We also view T_v as an algebraic subgroup (only well-defined up to conjugation) of G_ℓ in the following sense. Since v is unramified, we have an embedding $\text{Gal}(\bar{k}_v/k_v) \cong \text{Gal}(\bar{L}_v/L_v) \rightarrow \text{Gal}(\bar{K}/K)$ after choosing an embedding $\bar{K} \rightarrow \bar{L}_v$. Hence we view the Frobenius $Frob_v$ as an element of G_ℓ . Due to Katz and Messing [KM74], the characteristic polynomial of $\varphi_v^{m_v}$ acting on $H_{\text{cris}}^1(A_{k_v}/W(k_v))$ is the same⁵ as the characteristic polynomial of $Frob_v$ acting on $H_{\text{ét}}^1(A_{\bar{K}}, \mathbb{Q}_\ell)$. Hence T_v is isomorphic to the algebraic group generated by semi-simple element $Frob_v$ in G_ℓ . From now on, when we view T_v as a subgroup of G_ℓ , we identify T_v with the group generated by $Frob_v$.

Here are some important properties of Frobenius tori.

Theorem 3.1.5 (Serre, see also [Chi92, Cor. 3.8]). *There is a set M_{\max} of finite places of $K^{\text{ét}}$ of natural density one and disjoint from Σ such that for any $v \in M_{\max}$, the algebraic group T_v is connected and it is a maximal torus of G_ℓ .*

Proposition 3.1.6 ([Chi92, Prop. 3.6 (b)]). *For L large enough (for instance, containing all the n -torsion points for some $n \geq 3$), all but finitely many T_v 's are connected.*

⁵To compare the two polynomials, we notice that both of them have \mathbb{Z} -coefficients.

Corollary 3.1.7. *For L large enough, the relative de Rham–Tate group G^L is connected and $G^L = G^{L'}$ for $L \subset L'$.*

Proof. Let $(G^L)^\circ$ be the connected component of G^L . It is reductive and φ_σ -stable for all archimedean places σ . By Proposition 3.1.6, for all but finitely many v , the group T_v is connected and hence is contained in $(G^L)^\circ_{L_v}$. Therefore, $\varphi_v^{m_v} \in T_v(L_v) \subset (G^L)^\circ(L_v)$ and $(G^L)^\circ = G^L$ by Lemma 2.4.4(4). Let v' be a place of L' over v . By definition, $T_{v'}$ is a subgroup of T_v is finite index. Since T_v is connected, we have $\phi_v^{m_v} \in T_v = T_{v'} \subset G^{L'}$. We conclude by Lemma 2.4.4(4). \square

Remark 3.1.8. One reason to introduce the notion of relative de Rham–Tate cycles is that G^L behaves like G_ℓ in the sense that both of them become connected if one replace the base field K by large enough L .

The following lemma is of its own interest.

Lemma 3.1.9. *The number field K^{dR} is contained in $K^{\text{ét}}$.*

Proof. For the simplicity of notation, we enlarge K^{dR} to contain $K^{\text{ét}}$ and prove that they are equal. Let v be a finite place of $K^{\text{ét}}$ above p such that p splits completely in $K^{\text{ét}}/\mathbb{Q}$ and we identify $K_v^{\text{ét}}$ with \mathbb{Q}_p via v . Let w be a place of K^{dR} above v . Denote by σ the Frobenius in $\text{Gal}(K_w^{\text{dR}}/\mathbb{Q}_p) = \text{Gal}(K_w^{\text{dR}}/K_v^{\text{ét}})$. We consider the algebraic group T_v generated by $\varphi_v \in G_{\text{dR}}^{K_v^{\text{ét}}}(K_v^{\text{ét}})$. If $v \in M_{\max}$ as in Theorem 3.1.5, then T_v is connected and hence $T_v \subset G_{\text{dR}, K_v^{\text{ét}}}$. This implies that $\varphi_v \in G_{\text{dR}}(K_v^{\text{ét}})$. For any m, n , let $W' \subset (H_{\text{dR}}^1(A, K^{\text{dR}}))^{m, n}$ be the K^{dR} -linear span of all de Rham–Tate cycles in $(H_{\text{dR}}^1(A, K^{\text{dR}}))^{m, n}$. By 2.1.2, there exists a K -linear subspace W of $(H_{\text{dR}}^1(A, K))^{m, n}$ such that $W' = W \otimes K^{\text{dR}}$. Since G_{dR} acts trivially on W' and W , the Frobenius φ_v acts on $W \otimes_K K_v^{\text{ét}}$ trivially and ϕ_w acts on $W' \otimes_{K^{\text{dR}}} K_w^{\text{dR}}$ as the σ -linearly extension of φ_v . Hence the elements in W' that are stabilized by φ_w are contained in $W \otimes_K K_v^{\text{ét}}$. That is to say that all de Rham–Tate cycles are defined over $K_v^{\text{ét}}$. As m, n are arbitrary, we have $K_w^{\text{dR}} = K_v^{\text{ét}}$. This implies that p splits completely in K^{dR}/\mathbb{Q} and hence $K^{\text{dR}} = K^{\text{ét}}$ by the Chebotarev density theorem. \square

Remark 3.1.10. From Theorem 2.3.7, we see the definition field of a de Rham–Tate cycle induced from an endomorphism of $A_{\bar{K}}$ is the same as the definition field of this endomorphism. Hence K^{dR} contains the definition field of all endomorphisms. Then K^{dR} and $K^{\text{ét}}$ are the same if the definition field of all endomorphisms is $K^{\text{ét}}$. This is the case when one can choose a set of ℓ -adic Tate cycles all induced from endomorphisms of A to cut out G_ℓ .

Now we discuss some refinements of Theorem 3.1.5 and Proposition 3.1.6. In the rest of this subsection, the definition field K of the polarized abelian variety A is always assumed to be Galois over \mathbb{Q} . The main result is:

Proposition 3.1.11. *Assume that $G_\ell^\circ(A) = \text{GSp}_{2g}(\mathbb{Q}_\ell)$. Then there exists a set M of rational primes with natural density one such that for any $p \in M$ and any finite place v of K lying over p , the algebraic group T_v generated by $\varphi_v^{m_v}$ (where $m_v = [K_v : \mathbb{Q}_p]$) is of maximal rank. In particular, T_v is connected for such v .⁶*

⁶Our proof is a direct generalization of the proof of Theorem 3.1.5 by Serre. Will Sawin pointed out to me that it is possible to prove this proposition by applying Chavdarov’s method ([Cha97]) to $\text{Res}_{\mathbb{Q}}^K A$.

The idea of the proof is to apply the Chebotarev density theorem to a suitably chosen Zariski closed subset of the ℓ -adic monodromy group of B , the Weil restriction $\text{Res}_{\mathbb{Q}}^K A$ of A . As we are in characteristic zero, the scheme B is an abelian variety over \mathbb{Q} . We have $B_K = \prod_{\sigma \in \text{Gal}(K/\mathbb{Q})} A^\sigma$, where $A^\sigma = A \otimes_{K, \sigma} K$. It is a standard fact that $B^\vee = \text{Res}_{\mathbb{Q}}^K A^\vee$ and hence the polarization on A induces a polarization on B over \mathbb{Q} . Moreover, the polarization on A induces a polarization on A^σ . The map $\sigma : A(\bar{K}) \rightarrow A^\sigma(\bar{K}), P \mapsto \sigma(P)$ induces a map on Tate modules $\sigma : T_\ell(A) \rightarrow T_\ell(A^\sigma)$. This map is an isomorphism between \mathbb{Z}_ℓ -modules.

Lemma 3.1.12. *The map $\sigma : T_\ell(A) \rightarrow T_\ell(A^\sigma)$ induces an isomorphism between the ℓ -adic monodromy groups $G_\ell(A)$ and $G_\ell(A^\sigma)$.*

Proof. Via σ , the image of $\text{Gal}(\bar{K}/K)$ in $\text{End}(T_\ell(A^\sigma))$ is identified as that of $\text{Gal}(\bar{K}/K)$ in $\text{End}(T_\ell(A))$. Hence $G_\ell(A) \simeq G_\ell(A^\sigma)$ as $T_\ell(-)^\vee = H_{\text{ét}}^1((-)_{\bar{K}}, \mathbb{Q}_\ell)$. \square

We start from the following special case to illustrate the idea of the proof of Proposition 3.1.11.

Proposition 3.1.13. *Assume that $G_\ell(A) = \text{GSp}_{2g, \mathbb{Q}_\ell}$ and that A^σ is not geometrically isogenous to A^τ for any distinguished $\sigma, \tau \in \text{Gal}(K/\mathbb{Q})$. Then there exists a set M of rational primes with natural density 1 such that for any $p \in M$ and any $v|p$, the group T_v is of maximal rank. That is, the rank of T_v equals to the rank of $G_\ell(A)$. In particular, T_v is connected for such v .*

Proof. We use the same idea as in the proof of Theorem 3.1.5 by Serre. His idea is to first construct a proper Zariski closed subvariety $Z \subset G_\ell(A)$ as follows (see also [Chi92, Thm. 3.7]) and then to apply the Chebotarev density theorem:

- (1) Z is invariant under conjugation by $G_\ell(A)$,
- (2) if $u \in G_\ell(A)(\mathbb{Q}_\ell) \setminus Z(\mathbb{Q}_\ell)$ semisimple, then the algebraic subgroup of G_ℓ generated by u is of maximal rank.

Since $G_\ell(A)$ is connected, $Z(\mathbb{Q}_\ell)$ is of measure zero in $G_\ell(A)(\mathbb{Q}_\ell)$ with respect to the usual Haar measure. We will define a Zariski closed subset $W \subset G_\ell(B)$ which has similar properties as Z .

Let $G_\ell^K(B)$ be the Zariski closure of $\text{Gal}(\bar{K}/K)$ in $\text{GL}(H_{\text{ét}}^1(B_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell))$. Via the isomorphism $H_{\text{ét}}^1(B_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) \cong \oplus H_{\text{ét}}^1(A_{\bar{K}}^\sigma, \mathbb{Q}_\ell)$ of $\text{Gal}(\bar{K}/K)$ -modules, we view $G_\ell^K(B)$ as a subgroup of $\prod G_\ell(A^\sigma)$. By the assumption that A^σ 's are not geometrically isogenous to each other and [Lom15, Thm. 4.1, Rem. 4.3], we have $G_\ell^K(B) = \mathbb{G}_m \cdot \prod SG_\ell(A^\sigma)$, where $SG_\ell \subset G_\ell$ is the subgroup of elements with determinant 1. Indeed, $\text{Lie } SG_\ell(A^\sigma) = \mathfrak{sp}_{2g, \mathbb{Q}_\ell}$ of type C and the representations are all standard representations and then Rem. 4.3 in *loc. cit.* verified that Lombardo's theorem is applicable in our situation. Then By Lemma 3.1.12, we have $G_\ell^K(B) \simeq \mathbb{G}_m \cdot SG_\ell(A)^{[K:\mathbb{Q}]}$. This is the neutral connected component of $G_\ell(B)$.

The map $\text{Gal}(\mathbb{Q}/\mathbb{Q}) \rightarrow G_\ell(B)(\mathbb{Q}_\ell) \rightarrow G_\ell(B)(\mathbb{Q}_\ell)/G_\ell^K(B)(\mathbb{Q}_\ell)$ induces a surjection $\text{Gal}(K/\mathbb{Q}) \twoheadrightarrow G_\ell(B)(\mathbb{Q}_\ell)/G_\ell^K(B)(\mathbb{Q}_\ell)$. Given $\sigma \in \text{Gal}(K/\mathbb{Q})$, we denote by $\sigma G_\ell^K(B)$ the subvariety of $G_\ell(B)$ corresponding to the image of σ in the above map. Let m be the order of $\sigma \in \text{Gal}(K/\mathbb{Q})$. We consider those p unramified in K/\mathbb{Q} whose corresponding Frobenii in $\text{Gal}(K/\mathbb{Q})$ fall into c_σ , the conjugacy class of σ . We have $m_v = m$ for all $v|p$.

Consider the composite map $m_\tau : \sigma G_\ell^K(B) \rightarrow G_\ell^K(B) \rightarrow G_\ell(A^\tau) \simeq G_\ell(A)$, where the first map is defined by $g \mapsto g^m$ and the second map is the natural

projection. Let $W_{\sigma,\tau}$ be the preimage of Z and W_σ be $\cup_{\tau \in \text{Gal}(K/\mathbb{Q})} W_{\sigma,\tau}$. Since by definition W_σ is a proper Zariski subvariety of the connected variety $\sigma G_\ell^K(B)$, the measure of $W_\sigma(\mathbb{Q}_\ell)$ is zero.

Claim. If the Frobenius $Frob_p$ (well-defined up to conjugacy) is not contained in the conjugacy invariant set $\cup_{\gamma \in c_\sigma} W_\gamma(\mathbb{Q}_\ell)$, then for any $v|p$, the algebraic subgroup $T_v \subset G_\ell(A)$ is of maximal rank.

Proof. The subvariety $\cup_{\gamma \in c_\sigma} W_\gamma$ is invariant under the conjugation of $G_\ell^K(B)$ because Z is invariant under the conjugation of $G_\ell(A)$. This subvariety is moreover conjugation invariant under the action $G_\ell(B)$ since $\tau W_\sigma \tau^{-1} = W_{\tau\sigma\tau^{-1}}$ by definition. By second property of Z and the definition of the map m_τ , we see that the image of $Frob_p^m$ generates a maximal torus in $G_\ell(A^\tau)$. For each $v|p$, the Frobenius $Frob_v$ is the image of $Frob_p^m$ in $G_\ell(A^\tau)$ for some τ and hence T_v is of maximal rank. \square

Let W be $\cup_\sigma W_\sigma$. It is invariant under the conjugation of $G_\ell(B)$. As each $W_{\sigma,\tau}(\mathbb{Q}_\ell)$ is of measure zero in $G_\ell(B)(\mathbb{Q}_\ell)$, so is $W(\mathbb{Q}_\ell)$. By the Chebotarev density theorem (see for example [Ser12, Sec. 6.2.1]), we conclude that there exists a set M of rational primes with natural density 1 such that $Frob_p \notin W(\mathbb{Q}_\ell)$. Then the proposition follows from the above claim. \square

Remark 3.1.14. The assumption that $G_\ell(A) = \text{GSp}_{2g,\mathbb{Q}_\ell}$ can be weakened. The proof still works if one has $G_\ell^K(B) = \mathbb{G}_m \cdot \prod SG_\ell(A^\sigma)$. In other words, the proposition holds true whenever [Lom15, Thm. 4.1, Rem. 4.3] is applicable. For example, when A has odd dimension and is not of type IV in Albert's classification.

The following property of GSp_{2g} is used in an essential way of our proof of Proposition 3.1.11. It is well-known, but we give a proof for the sake of completeness.

Lemma 3.1.15. *If G is an algebraic subgroup of $\text{GL}(H_{\text{ét}}^1(A_{\bar{K}}, \mathbb{Q}_\ell))$ containing $\text{GSp}_{2g,\mathbb{Q}_\ell}$ as a normal subgroup, then $G = \text{GSp}_{2g,\mathbb{Q}_\ell}$. In particular, $G_\ell^\circ(A) = \text{GSp}_{2g,\mathbb{Q}_\ell}$ implies that $G_\ell(A)$ is connected.*

Proof. Let g be a $\overline{\mathbb{Q}_\ell}$ -point of G . Then $\text{ad}(g)$ induces an automorphism of $\text{GSp}_{2g,\overline{\mathbb{Q}_\ell}}$ by the assumption that $\text{GSp}_{2g,\mathbb{Q}_\ell}$ is a normal subgroup. As $\text{ad}(g)$ preserves determinant, we view $\text{ad}(g)$ as an automorphism of $\text{Sp}_{2g,\overline{\mathbb{Q}_\ell}}$. Since $\text{Sp}_{2g,\overline{\mathbb{Q}_\ell}}$ is a connected, simply connected linear algebra group whose Dynkin diagram does not have any nontrivial automorphism, any automorphism of $\text{Sp}_{2g,\overline{\mathbb{Q}_\ell}}$ is inner. Hence $\text{ad}(g) = \text{ad}(h)$ for some $\overline{\mathbb{Q}_\ell}$ -point h of $\text{Sp}_{2g,\mathbb{Q}_\ell}$. Then g and h are different by an element in the centralizer of $\text{Sp}_{2g,\overline{\mathbb{Q}_\ell}}$ in $\text{GL}_{2g,\overline{\mathbb{Q}_\ell}}$. Since the centralizer is \mathbb{G}_m , we conclude that g is in $\text{GSp}_{2g,\mathbb{Q}_\ell}(\overline{\mathbb{Q}_\ell})$. \square

Proof of Proposition 3.1.11. As in the proof of Proposition 3.1.13, it suffices to construct a Zariski closed set $W \subset G_\ell(B)$ such that

- (1) $W(\mathbb{Q}_\ell)$ is of measure zero with respect to the Haar measure on $G_\ell(B)(\mathbb{Q}_\ell)$.
- (2) W is invariant under conjugation by $G_\ell(B)$,
- (3) if $u \in G_\ell(B)(\mathbb{Q}_\ell) \setminus W(\mathbb{Q}_\ell)$ semisimple, then the algebraic subgroup of $G_\ell(B)$ generated by u is of maximal rank.

We first show that, to construct such W , it suffices to construct $W_\sigma \subset \sigma G_\ell^K(B)$ for each $\sigma \in \text{Gal}(K/\mathbb{Q})$ such that

- (1) $W_\sigma(\mathbb{Q}_\ell)$ is of measure zero with respect to the Haar measure on $G_\ell(B)(\mathbb{Q}_\ell)$.
- (2) W_σ is invariant under conjugation by $G_\ell^K(B)$.
- (3) if $u \in \sigma G_\ell^K(B)(\mathbb{Q}_\ell) \setminus W(\mathbb{Q}_\ell)$ semisimple, then the algebraic subgroup of $G_\ell(B)$ generated by u is of maximal rank.

Indeed, given such W_σ , we define W' to be $\cup_\sigma W_\sigma$. This set satisfies (1) and (3) and is invariant under conjugation by $G_\ell^K(B)$. We then define W to be the $G_\ell(B)$ -conjugation invariant set generated by W' . Since $[G_\ell(B) : G_\ell^K(B)]$ is finite, W as a set is a union of finite copies of W' and hence satisfies (1) and (3).

To construct W_σ , let $C \subset \text{Gal}(K/\mathbb{Q})$ be the subgroup generated by σ . Consider $\{A^\tau\}_{\tau \in C}$. We have a partition $C = \sqcup_{1 \leq i \leq r} C_i$ with respect to the \bar{K} -isogeny classes of A^τ . These C_i have the same cardinality m/r . For any $\alpha \in \text{Gal}(K/\mathbb{Q})$, the partition of $\alpha C = \sqcup \alpha C_i$ gives the partition of $\{A^\tau\}_{\tau \in \alpha C}$ with respect to the \bar{K} -isogeny classes.

Consider the map $m_\alpha : \sigma G_\ell^K(B) \rightarrow G_\ell^K(B) \rightarrow G_\ell(A^\alpha) \simeq G_\ell(A)$ and define $W_{\sigma,\alpha}$ to be the preimage of Z and W_σ to be $\cup_{\alpha \in \text{Gal}(K/\mathbb{Q})} W_{\sigma,\alpha}$ as in the proof of Proposition 3.1.13. The proof of the claim there shows that W_σ satisfies (2) and (3).

Now we focus on (1). By the assumption and Lemma 3.1.15, the group $G_\ell(A)$ is connected and hence $Z(\mathbb{Q}_\ell)$ is of measure zero. Let γ be σ^r . Consider $r : \sigma G_\ell^K(B) \rightarrow \gamma G_\ell^K(B)$ defined by $g \mapsto g^r$ and the composite map

$$(m/r)_\alpha : \gamma G_\ell^K(B) \rightarrow G_\ell^K(B) \rightarrow G_\ell(A^\alpha) \simeq G_\ell(A),$$

where the first map is defined by $g \mapsto g^{m/r}$ and the second map natural projection. Then $m_\alpha = (m/r)_\alpha \circ r$. Let W_r be $(m/r)_\alpha^{-1}(Z)$. Then $W_{\sigma,\alpha} = r^{-1}(W_r)$. Since any two of $\{A^\tau\}_{\tau=\alpha, \alpha\sigma, \dots, \alpha\sigma^{r-1}}$ are not geometrically isogenous, the same argument as in the proof of Proposition 3.1.13 shows that if $W_r(\mathbb{Q}_\ell)$ is of measure zero, so is $W_{\sigma,\alpha}(\mathbb{Q}_\ell)$.

Notice that $G_\ell^\circ(B_K) = \mathbb{G}_m \cdot \prod_{\sigma \in \mathbb{I}} S G_\ell(A^\sigma) = \mathbb{G}_m \cdot \text{Sp}_{2g}^{\mathbb{I}}$, where \mathbb{I} is a set of representatives of all isogeny classes in $\{A^\sigma\}_{\sigma \in \text{Gal}(K/\mathbb{Q})}$. By the facts that the centralizer of GSp_{2g} in GL_{2g} is \mathbb{G}_m and that $G_\ell^\circ(B)$ is a normal subgroup of $G_\ell(B)$, the map $(m/r)_\alpha$ is up to a constant the same as the following map:

$$\gamma G_\ell^K(B) \rightarrow \text{Hom}_{\mathbb{Q}_\ell}(H_{\text{ét}}^1(A^{\alpha\gamma}), H_{\text{ét}}^1(A^\alpha))^\times \cong \text{GL}(H_{\text{ét}}^1(A^\alpha)) \rightarrow \text{GL}(H_{\text{ét}}^1(A^\alpha)),$$

where the first map is the natural projection, the middle isomorphism is given by a chosen isogeny between A^α and $A^{\alpha\gamma}$, and the last map is $g \mapsto g^{m/r}$.

The fact that $\gamma G_\ell^K(B)$ normalizes $G_\ell^\circ(B_K)$ allows us to apply Lemma 3.1.15 to the image of the above map and see that the above map factors through

$$\text{GSp}(H_{\text{ét}}^1(A^\alpha)) \rightarrow \text{GSp}(H_{\text{ét}}^1(A^\alpha)), g \mapsto g^{m/r}$$

and hence $W_{\sigma,\alpha}(\mathbb{Q}_\ell)$ is of measure zero. \square

3.2. The Mumford–Tate conjecture and the proofs of Theorem 1.1 and Theorem 1.2.

Conjecture 3.2.1 (Mumford–Tate). *For any rational prime ℓ , we have $G_\ell^\circ = G_{\text{MT}} \otimes \mathbb{Q}_\ell$ via the comparison isomorphism between the étale and the Betti cohomologies.*

Lemma 3.2.2. *If 3.2.1 holds for the abelian variety A , then the reductive groups G_ℓ , G_ℓ^L , G_{dR} , and G_{MT} have the same rank.*

Proof. 3.2.1 implies that G_ℓ and G_{MT} have the same rank. Then by Theorem 3.1.5, there are infinitely many finite places v such that the Frobenius torus T_v is a maximal torus of G_{MT} . Since T_v is a subtorus of G^L except for finitely many v , we have that G^L and hence G_{dR} have the same rank as G_{MT} by Corollary 2.2.5. \square

The assertion of the above lemma is equivalent to the Mumford–Tate conjecture by the following lemma due to Zarhin and the Faltings isogeny theorem (see for example [Vas08, Sec. 1.1]).

Lemma 3.2.3 ([Zar92, Sec. 5, key lemma]). *Let V be a vector space over a field of characteristic zero and $H \subset G \subset \text{GL}(V)$ be connected reductive groups. Assume that H and G have the same rank and the same centralizer in $\text{End}(V)$. Then $H = G$.*

Using this lemma, we prove a special case of 2.1.4.

Theorem 3.2.4. *Assume that the polarized abelian variety A is defined over \mathbb{Q} and that $G_\ell(A)$ is connected. Then the centralizer of G_{dR}° in $\text{End}(H_{\text{dR}}^1(A, \mathbb{Q}))$ is $\text{End}(A) \otimes \mathbb{Q}$ and moreover, 2.1.4 holds if 3.2.1 holds.*

Proof. The assumption is equivalent to that $K^{\text{ét}} = \mathbb{Q}$. Then by Theorem 3.1.5, we see that T_p is connected for a density one set of rational primes p . Therefore, for such p , the Frobenius $\varphi_p \in T_p(\mathbb{Q}_p) \subset G_{\text{dR}}^\circ(\mathbb{Q}_p)$ and any s lying in the centralizer of G_{dR}° is fixed by φ_p . In other words, s is a 1-de Rham–Tate cycle and by Theorem 2.3.7, $s \in \text{End}^\circ(A)$. The second assertion follows directly from Lemma 3.2.2 and Lemma 3.2.3. \square

Similar to [Pin98], although 2.1.4 has not been proved in full, we show that the de Rham–Tate group G_{dR} is of a very restricted form when we assume that A is defined over \mathbb{Q} and that $K^{\text{ét}} = \mathbb{Q}$. We need the following definition to state our result.

Definition 3.2.5 ([Pin98, Def. 4.1]). A *strong Mumford–Tate pair* (of weight $\{0, 1\}$) over K is a pair (G, ρ) of a reductive algebraic group over K and a finite dimension faithful algebraic representation of G over K such that there exists a cocharacter $\mu : \mathbb{G}_{m, \bar{K}} \rightarrow G_{\bar{K}}$ satisfying:

- (1) the weights of $\rho \circ \mu$ are in $\{0, 1\}$,
- (2) $G_{\bar{K}}$ is generated by $G(\bar{K}) \rtimes \text{Gal}(\bar{K}/K)$ -conjugates of μ .

We refer the reader to [Pin98, Sec. 4, especially Table 4.6, Prop. 4.7] for the list of strong Mumford–Tate pairs.

Theorem 3.2.6. *If the polarized abelian variety A is defined over \mathbb{Q} and $G_\ell(A)$ is connected, then there exists a normal subgroup G of G_{dR}° defined over \mathbb{Q} such that*

- (1) (G, ρ) is a strong Mumford–Tate pair over \mathbb{Q} , where ρ is the tautological representation $\rho : G \subset G_{\text{dR}} \rightarrow \text{GL}(H_{\text{dR}}^1(A, \mathbb{Q}))$,
- (2) The centralizer of G in $\text{End}(H_{\text{dR}}^1(A, \mathbb{Q}))$ is $\text{End}^\circ(A)$.

If we further assume that $\text{End}(A)$ is commutative, then we can take G to be G_{dR}° .

To construct the cocharacter μ , we need the following results.

Definition 3.2.7. Due to Wintenberger (see also [Pin98, Sec. 2]), every weakly admissible filtered φ -module over L_v has a canonical splitting. For any finite $v \notin \Sigma$, the cocharacter over L_v of the canonical splitting of the filtration $\Gamma(A, \Omega_{A/K}) \otimes L_v \subset H_{\text{dR}}^1(A, L_v)$ is defined to be the (v -adic) Hodge cocharacter μ_v .

Lemma 3.2.8. *The v -adic Hodge cocharacter does not depend on the choice of L containing K . It factors through $G^L(L_v)$ and hence through $(G_{\text{dR}}(K_v))^\circ$. Moreover, given an embedding $K_v \rightarrow \mathbb{C}$ extending $\sigma : K \rightarrow \mathbb{C}$, then the Hodge cocharacter μ_σ defined by Hodge decomposition (see 2.2.4) is conjugate to μ_v over $G^L(\mathbb{C})$.*

Proof. All but the last assertion are direct consequences of the definition. To prove the last assertion, notice that the stabilizers of μ_σ and μ_v in G^L are Levi subgroups in the same parabolic subgroup, the stabilizer of the Hodge filtration in $G_{\mathbb{C}}^L$. Since such two Levi subgroups are conjugate by an element g in $G^L(\mathbb{C})$ and μ_σ, μ_v are the same on the graded pieces, these two characters are conjugate by g . \square

Lemma 3.2.9. *There exists a cocharacter $\mu : \mathbb{G}_{m, \bar{K}} \rightarrow G_{\bar{K}}$ such that after fixing an embedding $\bar{K} \rightarrow \bar{K}_v$ (resp. $\bar{K} \rightarrow \mathbb{C}$), the Hodge cocharacter μ_v (resp. μ_σ) is conjugate to μ over $G^L(\bar{K}_v)$ (resp. $G^L(\mathbb{C})$).*

Proof. One can always choose a \bar{K} -splitting so that its corresponding cocharacter factors through G^L (due to the existence of a Levi subgroup over \bar{K}). It is conjugate to μ_v and μ_σ by the same argument as in the proof of Lemma 3.2.8. \square

Proof of Theorem 3.2.6. Let μ be some cocharacter constructed in Lemma 3.2.9 and G be the smallest normal \mathbb{Q} -subgroup of G_{dR}° such that $G(\bar{\mathbb{Q}})$ contains the image of μ . Notice that different choices of μ are conjugate to each other over $\bar{\mathbb{Q}}$ and hence the definition of G is independent of the choice of μ .

The weights of $\rho \circ \mu$ are 0 or 1 since that is the case for $\rho \circ \mu_v$. Since the subgroup of G generated by $G_{\text{dR}}^\circ(\bar{\mathbb{Q}}) \rtimes \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -conjugates of μ must be defined over \mathbb{Q} and normal in G_{dR}° , this subgroup coincides with G . Since G_{dR}° is connected and reductive and the image of μ is contained in G , the set of $G_{\text{dR}}^\circ(\bar{\mathbb{Q}})$ -conjugates of μ is the same as the set of $G(\bar{\mathbb{Q}})$ -conjugates of μ . Hence (G, ρ) is a strong Mumford–Tate pair over \mathbb{Q} .

To show (2), by Theorem 2.3.7, it suffices to show that $\varphi_p \in G(\mathbb{Q}_p)$ for p in a set of natural density 1. By Theorem 3.1.5, it suffices to show that for any $p \in M_{\max}$, there exists an integer n_p ⁷ such that $\varphi_p^{n_p} \in G(\mathbb{Q}_p)$. Let $W \subset (H_{\text{dR}}^1(A, \mathbb{Q}))^{m,n}$ be the largest \mathbb{Q} -sub vector space with trivial G -action. Since G is normal in G_{dR}° , the group G_{dR}° acts on W . Then for all $p \in M_{\max}$, we have $\varphi_p \in G_{\text{dR}}^\circ(\mathbb{Q}_p)$ acts on $W \otimes \mathbb{Q}_p$. Since G is reductive, it can be defined to be the subgroup of $\text{GL}(H_{\text{dR}}^1(A, \mathbb{Q}))$ acting trivially on finitely many such W . Since $\varphi_v \in G_{\text{dR}}^\circ(\mathbb{Q}_p)$ is semi-simple, in order to show that $\varphi_p^{n_p} \in G(\mathbb{Q}_p)$, it suffices to prove that the eigenvalues of φ_p acting on $W \otimes \mathbb{Q}_p$ are all roots of unity.

Since $W \subset (H_{\text{dR}}^1(A, \mathbb{Q}))^{m,n}$, the eigenvalues of φ_p are all algebraic numbers. Since Frob_p acts on $(H_{\text{ét}}^1(A_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell))^{m,n(i)}$ with all eigenvalues being ℓ -adic units for $\ell \neq p$, the eigenvalues of φ_p are also ℓ -adic units. Now we show that these eigenvalues are p -adic units. Let H_p be the Tannakian fundamental group of the abelian tensor category generated by sub weakly admissible filtered φ -modules of $(H_{\text{cris}}^1(A/W(\mathbb{F}_p)) \otimes \mathbb{Q}_p)^{m,n}$. For $p \in M_{\max}$, by [Pin98, Prop. 3.13], H_p is connected. By Lemma 2.1.3, every de Rham–Tate cycle generates a trivial filtered φ -module. Then by the definition of H_p , $H_p(\mathbb{Q}_p) \subset G_{\text{dR}}(\mathbb{Q}_p)$ and then $H_p(\mathbb{Q}_p) \subset G_{\text{dR}}^\circ(\mathbb{Q}_p)$. Hence $W \otimes \mathbb{Q}_p$ is a H_p -representation and then by the Tannakian equivalence, the filtered φ -module $W \otimes \mathbb{Q}_p$ is weakly admissible. By Lemma 3.2.8, μ_p acts on $W \otimes \mathbb{Q}_p$ trivially. Then the Newton cocharacter is also trivial. In other words, the

⁷By standard arguments, one can choose an n independent of p .

eigenvalues of φ_p are p -adic units. By the Weil conjecture, the archimedean norms of the eigenvalues are $p^{(m-n)/2}$. Then by the product formula, the weight $\frac{m-n}{2}$ must be zero and all the eigenvalues are roots of unity.

We now prove the last assertion. If $G \neq G_{\text{dR}}^\circ$, then we have an almost direct decomposition $G_{\text{dR}}^\circ = GH$ where H is some nontrivial normal connected subgroup of G_{dR}° commuting with G . Then H is contained in the centralizer of G and by (2), we have $H \subset \text{End}^\circ(A)$. By the assumption on $\text{End}^\circ(A)$, we see that H is commutative and hence $H \subset Z^\circ(G_{\text{dR}}^\circ)$. We draw a contradiction by showing that $Z^\circ(G_{\text{dR}}^\circ) \subset G$. By Theorem 3.2.4, we have

$$Z(G_{\text{dR}}^\circ) = G_{\text{dR}}^\circ \cap \text{End}^\circ(A) \subset G_{\text{MT}} \cap \text{End}^\circ(A) = Z(G_{\text{MT}}),$$

and hence $Z^\circ(G_{\text{dR}}^\circ) \subset Z^\circ(G_{\text{MT}})$. On the other hand, for all $p \in M_{\text{max}}$, the torus $T_p \subset G$. Hence we only need to show that $Z^\circ(G_{\text{MT}}) \subset T_p$. Since this statement is equivalent up to conjugation, we only need to show that $Z^\circ(G_{\text{MT}} \otimes \mathbb{Q}_\ell) \subset T_p \subset \text{GL}(H_{\text{et}}^1(A_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell))$. Since T_p is a maximal torus, we have $T_p \supset Z^\circ(G_\ell)$. We then conclude by [Vas08, Thm. 1.2.1] asserting that $Z^\circ(G_{\text{MT}} \otimes \mathbb{Q}_\ell) = Z^\circ(G_\ell)$. \square

3.3. A result of Noot and its consequence. It is well-known that the Mumford–Tate group is a torus if and only if A has complex multiplication. In particular, G_{dR}° is a torus when A has complex multiplication. In the rest of this section, we will show that the converse is also true.

Lemma 3.3.1. *If G^L commutes with μ_σ for some σ or equivalently commutes with μ_v for some v , then A has ordinary reduction at a positive density of primes of degree one (that is, splitting completely over \mathbb{Q}).*

Proof. The equivalence of the assumptions is due to Lemma 3.2.8. Let v be a finite place of K with residue characteristic p and assume that p splits completely in K/\mathbb{Q} . Then $K_v \cong \mathbb{Q}_p$. As before, let H_v be the Tannakian fundamental group of the abelian tensor category generated by sub weakly admissible filtered φ -modules of $(H_{\text{cris}}^1(A/W(k_v)) \otimes K_v)^{m,n}$. When $v \notin \Sigma$, we have $H_v \subset G^L$ by Lemma 2.4.4 (2). Let ν_v be the Newton (quasi-)cocharacter and fix a maximal torus $T \subset H_v$. As in [Pin98, Sec. 1 and 2], we define S_{μ_v} (resp. S_{ν_v}) to be the set of $H_v(\bar{K}_v) \rtimes \text{Gal}(\bar{K}_v/K_v)$ -conjugates of μ_v (resp. ν_v) factoring through $T(\bar{K}_v)$. Since all the $G^L(\bar{K}_v)$ -conjugacy of μ_v is itself and that μ_v is defined over K_v , we have $S_{\mu_v} = \{\mu_v\}$. By the weakly admissibility, we have that S_{ν_v} is contained in the convex polygon generated by S_{μ_v} (see [Pin98, Thm. 1.3, Thm. 2.3]). Hence $S_{\nu_v} = S_{\mu_v}$. Then we conclude that A has ordinary reduction at v by [Pin98, Thm. 1.5] if $v \notin \Sigma$. \square

Proposition 3.3.2. *If G^L commutes with μ_σ or μ_v for some σ or v , then A has complex multiplication and hence 2.1.4 holds for A .*

Corollary 3.3.3. *The assumption of Proposition 3.3.2 is satisfied when $(G_{\text{dR}})^\circ$ commutes with either μ_σ or μ_v . In particular, A has complex multiplication if and only if $(G_{\text{dR}})^\circ$ is a torus.*

3.3.4. To prove Proposition 3.3.2, we need following theorem due to Noot. Let $\{t_\alpha\}$ be a finite set of relative de Rham–Tate cycles over L such that G^L is the stabilizer of $\{t_\alpha\}$. Let v be a finite place of L with residue characteristic p such that $L_v \cong \mathbb{Q}_p$. We assume that $v \notin \Sigma$ and that A_L has ordinary good reduction at v . The later assumption holds for infinitely many v under the assumption of Proposition 3.3.2

by Lemma 3.3.1. Since $m_v = 1$ in our situation, we have $\varphi_v(t_\alpha) = t_\alpha$. Moreover, $t_\alpha \in \text{Fil}^0(H_{\text{dR}}^1(A, L))^{m,n}$ by Lemma 2.4.4 (2). Hence t_α is a ‘Tate cycle’ in the sense of [Noo96]. In the formal deformation space of A_{k_v} , Noot defined the formal locus \mathcal{N} where the horizontal extensions of all t_α are still in $\text{Fil}^0(H_{\text{dR}}^1(A, L))^{m,n}$ (see [Noo96, Sec. 2] for details).

On the other hand, for any embedding $\sigma : L \rightarrow \mathbb{C}$, the relative de Rham–Tate group G^L , viewed as a subgroup of G_{MT} , is defined over \mathbb{R} . Hence G^L defines a sub Hermitian symmetric domain of that defined by G_{MT} . Under the assumption of Proposition 3.3.2, the sub Hermitian symmetric domain defined by G^L is zero dimensional.

Theorem 3.3.5 ([Noo96, Thm. 2.8]).⁸ *The formal locus \mathcal{N} is a translate of a formal torus by a torsion point. Moreover, the dimension of \mathcal{N} equals to the dimension of the Hermitian symmetric space defined by G^L .*

Proof of Proposition 3.3.2. By Theorem 3.3.5 and 3.3.4, A_{L_v} is a torsion point in the formal deformation space. By the Serre–Tate theory, a torsion point corresponds to an abelian variety with complex multiplication. Hence A has complex multiplication and the last assertion comes from 2.1.5. \square

4. PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 1.3. In section 4.1, we study the irreducible sub representations of G_{dR}° in $H_{\text{dR}}^1(A/K) \otimes \bar{K}$. This part is valid for most abelian varieties without assuming the Mumford–Tate conjecture. To do this, we focus on the crystalline Frobenii action. The result of Pink that G_ℓ with its tautological representation is a weak Mumford–Tate pair over \mathbb{Q}_ℓ provides information on étale Frobenii and hence information on crystalline Frobenii by a result of Noot (see 4.1.3) relating these two. In section 4.2, we use the results in section 4.1, Theorem 2.3.7 and Proposition 3.1.11 to show that under the assumptions of Theorem 1.3, the centralizer of G_{dR}° in $\text{End}(H_{\text{dR}}^1(A, \bar{K}))$ coincides with that of G_{dR} and then complete the proof of Theorem 1.3.

4.1. Group theoretical discussions.

4.1.1. Throughout this section, we assume that $(G_{\text{MT}}(A)_{\bar{\mathbb{Q}}})^{\text{der}}$ does not have any simple factor of type SO_{2k} for $k \geq 4$. This holds under the assumptions of Theorem 1.3. The reason for this assumption is that we will use a result of Noot on the conjugacy class of Frobenius to avoid the usage of the Mumford–Tate conjecture. It is likely that one can remove this assumption for all the results in this subsection with some extra work.

Let $\rho_{\text{dR}} : G_{\text{dR}} \rightarrow \text{GL}(H_{\text{dR}}^1(A, K))$ be the tautological algebraic representation given in 2.2.1. We denote by $\rho_{\bar{K}} : G_{\text{dR}, \bar{K}} \rightarrow \text{GL}(H_{\text{dR}}^1(A, \bar{K}))$ the representation on \bar{K} -points. Assume that the $G_{\text{dR}}(\bar{K})$ -representation $\rho_{\bar{K}} = \bigoplus_{i=1}^n \rho_{\bar{K}, i}$ and that the restricted representations $\rho_{\bar{K}, i}|_{G_{\text{dR}, \bar{K}}^\circ} = \bigoplus_{j=1}^{n_i} \rho_{\bar{K}, i, j}$, where $\rho_{\bar{K}, i}$ ’s (resp. $\rho_{\bar{K}, i, j}$ ’s) are irreducible representations of $G_{\text{dR}}(\bar{K})$ (resp. $G_{\text{dR}}^\circ(\bar{K})$). We denote the vector space of $\rho_{\bar{K}, i}$ (resp. $\rho_{\bar{K}, i, j}$) by V_i (resp. $V_{i, j}$).

⁸Ananth Shankar pointed out to me that one may use Kisin’s results in [Kis10, Sec. 1.5] and the property of canonical lifting to prove this result.

The following lemma reduces comparing the centralizers of G_{dR} and G_{dR}° to studying the irreducibility of V_i as representations of $G_{\text{dR}, \bar{K}}^\circ$.

Lemma 4.1.2. *If V_i and V_j are not isomorphic as $G_{\text{dR}, \bar{K}}$ -representations, then they are not isomorphic as $G_{\text{dR}, \bar{K}}^\circ$ -representations. In particular, if all V_i are irreducible representations of $G_{\text{dR}, \bar{K}}^\circ$, then G_{dR}° and G_{dR} have the same centralizer in $\text{End}(H_{\text{dR}}^1(A, \bar{K}))$.*

4.1.3. Before giving the proof of this lemma, we first explain how to use a result of Noot to translate problems on representations of G_{dR}° to problems on representations of G_ℓ . We fix an embedding $\bar{K} \rightarrow \bar{\mathbb{Q}}_\ell$. Since the de Rham and étale cohomologies can be viewed as fiber functors of the category of motives with absolute Hodge cycles, we have an isomorphism of representations of $(G_{\text{MT}})_{\bar{\mathbb{Q}}_\ell}$:

$$H_{\text{dR}}^1(A, \bar{K}) \otimes \bar{\mathbb{Q}}_\ell \simeq H_{\text{ét}}^1(A_{\bar{K}}, \mathbb{Q}_\ell) \otimes \bar{\mathbb{Q}}_\ell.$$

By Theorem 2.3.7, the left hand side, as a representation of $(G_{\text{MT}})_{\bar{\mathbb{Q}}_\ell}$, decomposes into irreducible ones $\oplus V_i \otimes \bar{\mathbb{Q}}_\ell$ and $V_i \otimes \bar{\mathbb{Q}}_\ell \cong V_j \otimes \bar{\mathbb{Q}}_\ell$ if and only if they are isomorphic as representations of G_{dR} . Via the above isomorphism, we denote by $V_i^{\text{ét}}$ the image of $V_i \otimes \bar{\mathbb{Q}}_\ell$. Then by Faltings isogeny theorem, $V_i^{\text{ét}}$ are irreducible representations of $(G_\ell)_{\bar{\mathbb{Q}}_\ell}$ and any two of them are isomorphic if and only if they are isomorphic as representations of $(G_{\text{MT}})_{\bar{\mathbb{Q}}_\ell}$.

Now we study the action of the Frobenius torus T_v° on both sides. More precisely, we use $(T_v^\circ)_{\bar{K}_v}$ to denote the base change of the crystalline one acting on the left hand side and use $(T_v^\circ)_{\bar{\mathbb{Q}}_\ell}$ to denote the base change of the étale one acting on the right hand side. By [Noo09, Thm. 4.2], after raising to high enough power, $\varphi_v^{m_v}$ is conjugate to Frob_v by an element of G_{MT} .⁹ Then the weights of the action of T_v° on V_i and $V_i^{\text{ét}}$ coincide and that V_i is isomorphic to V_j as representations of $(T_v^\circ)_{\bar{K}_v}$ is equivalent to that $V_i^{\text{ét}}$ is isomorphic to $V_j^{\text{ét}}$ as representations of $(T_v^\circ)_{\bar{\mathbb{Q}}_\ell}$.

Proof of Lemma 4.1.2. It suffices to show that $V_i \otimes \bar{K}_v$ and $V_j \otimes \bar{K}_v$ are not isomorphic as representations of the Frobenius torus $(T_v)_{\bar{K}_v}$ for some $v \in M_{\text{max}}$. Then the corresponding representations $V_i^{\text{ét}}$ and $V_j^{\text{ét}}$ of $(G_\ell)_{\bar{\mathbb{Q}}_\ell}$ are non-isomorphic. Since T_v is a maximal torus of G_ℓ , then $V_i^{\text{ét}}$ and $V_j^{\text{ét}}$ are non-isomorphic as representations of $(T_v)_{\bar{\mathbb{Q}}_\ell}$. Then by 4.1.3, we see that V_i and V_j are not isomorphic as representations of $(T_v)_{\bar{K}_v}$. \square

4.1.4. By construction, we have a fiber functor $\omega : \mathcal{M}_{\text{dRT}} \rightarrow \text{Vec}_{K^{\text{dR}}}$. In other words, ω is a fiber functor over $\text{Spec } K^{\text{dR}}$, viewed as a $\text{Spec } \mathbb{Q}$ -scheme. The functor $\underline{\text{Aut}}^\otimes(\omega)$ is representable by a $\text{Spec } K^{\text{dR}} / \text{Spec } \mathbb{Q}$ -groupoid \mathfrak{G} and \mathfrak{G} is faithfully flat over $\text{Spec } K^{\text{dR}} \times_{\text{Spec } \mathbb{Q}} \text{Spec } K^{\text{dR}}$ (see [Mil92, Thm. A.8] or [Del90, Thm. 1.12]).¹⁰ Let $v \notin \Sigma$ (defined in section 3) be a finite place of K^{dR} giving rise to an embedding $K^{\text{dR}} \rightarrow K_v^{\text{dR}}$. Let \mathfrak{G}_v be the $\text{Spec } K_v^{\text{dR}} / \text{Spec } \mathbb{Q}_p$ -groupoid obtained by

⁹More precisely, this means that after raising to high enough power, there exists an element $g \in G_{\text{MT}}(\bar{K})$ such that g is conjugate to $\varphi_v^{m_v}$ by some element in $G_{\text{MT}}(\bar{K}_v)$ and that g is conjugate to Frob_v by some element in $G_{\text{MT}}(\bar{\mathbb{Q}}_\ell)$.

¹⁰Here we use the language of groupoid. One may also view \mathfrak{G} as a Galois gerb in the sense of Langlands–Rapoport for the following reason. Since \mathfrak{G} is a torsor of a smooth algebraic group, it is trivial étale locally and hence $\mathfrak{G}(\bar{\mathbb{Q}}) \rightarrow (\text{Spec } \bar{\mathbb{Q}} \times_{\text{Spec } \mathbb{Q}} \text{Spec } \bar{\mathbb{Q}})(\bar{\mathbb{Q}})$ is surjective. We then have the exact sequence (see for example [Mil92, pp. 67])

$$1 \rightarrow G_{\text{dR}}(\bar{\mathbb{Q}}) \rightarrow \mathfrak{G}(\bar{\mathbb{Q}}) \rightarrow \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow 1.$$

base changing \mathfrak{G} to $\mathrm{Spec} K_v^{\mathrm{dR}} \times_{\mathrm{Spec} \mathbb{Q}_p} \mathrm{Spec} K_v^{\mathrm{dR}}$. Since $\varphi_v(s_\alpha) = s_\alpha$ for all de Rham–Tate cycles $\{s_\alpha\}$, the Frobenius semi-linear morphism $\varphi_v \in \mathfrak{G}_v(K_v^{\mathrm{dR}})$. Since \mathfrak{G} acts on G_{dR} by conjugation, the action $\mathrm{ad}(\varphi_v)$, the conjugation by φ_v , is an isomorphism between the neutral connected components $\sigma^* G_{\mathrm{dR}, K_v^{\mathrm{dR}}}$ and $G_{\mathrm{dR}, K_v^{\mathrm{dR}}}$, where $\sigma : K_v^{\mathrm{dR}} \rightarrow K_v^{\mathrm{dR}}$ is the Frobenius. In terms of K_v^{dR} -points, $\mathrm{ad}(\varphi_v)$ is a σ -linear automorphism of both $G_{\mathrm{dR}}(K_v^{\mathrm{dR}})$ and $G_{\mathrm{dR}}^\circ(K_v^{\mathrm{dR}})$.¹¹

Proposition 4.1.5. *Assume that $A_{\bar{K}}$ is simple. Then all $\rho_{\bar{K}, i, j}$ are of the same dimension. Moreover, if we further assume the assumption in 4.1.1 and that a maximal subfield of $\mathrm{End}_{\bar{K}}^\circ(A)$ is Galois over \mathbb{Q} or that $\mathrm{End}_{\bar{K}}^\circ(A)$ is a field, then there exists a choice of decomposition $\oplus V_i$ such that $\varphi_v(V_{i, j}) = V_{\sigma(i), \tau_{v, i}(j)}$, where σ is a permutation of $\{1, \dots, n\}$ and for each i , $\tau_{v, i}$ is a permutation of $\{1, \dots, n_i\}$.*

Proof. We fix a finite extension L of K such that all the $V_{i, j}$ are defined over L . Let $v \notin \Sigma$ be a place of L . As discussed in 4.1.4, $\mathrm{ad}(\varphi_v)$ preserves the set $G_{\mathrm{dR}}^\circ(L_v)$. Therefore, for any nonzero vector $v_{i, j} \in V_{i, j}$, as \mathbb{Q}_p -linear spaces,

$$\begin{aligned} \varphi_v(V_{i, j}) &= \varphi_v(\mathrm{Span}_{L_v}(G_{\mathrm{dR}}^\circ(L_v)(v_{i, j}))) = \mathrm{Span}_{L_v}(\varphi_v(G_{\mathrm{dR}}^\circ(L_v)(v_{i, j}))) \\ &= \mathrm{Span}_{L_v}(\mathrm{ad}(\varphi_v)(G_{\mathrm{dR}}^\circ(L_v)(\varphi_v(v_{i, j})))) = \mathrm{Span}_{L_v}(G_{\mathrm{dR}}^\circ(L_v)(\varphi_v(v_{i, j}))). \end{aligned}$$

In other words, as an L_v -vector space, $\varphi_v(V_{i, j})$ is the same as the space of the irreducible $G_{\mathrm{dR}}^\circ(L_v)$ -sub representation generated by $\varphi_v(v_{i, j})$. Similarly for V_i , we have that the vector space $\varphi_v(V_i)$ is the same as the vector space of an irreducible $G_{\mathrm{dR}}(L_v)$ -sub representation. In particular, $\varphi_v(V_{i, j})$ is contained in $\oplus_{\dim V_{k, l} = \dim V_{i, j}} V_{k, l}$. Let V' be $\oplus_{\dim V_{k, l} = \dim V_{i, j}} V_{k, l}$ and V'' be $\oplus_{\dim V_{k, l} \neq \dim V_{i, j}} V_{k, l}$. Then $\varphi_v(V') = V'$ and $\varphi_v(V'') = V''$. Let pr' be the projection to V' . Then $\varphi_v(pr') = pr'$ for all $v \notin \Sigma$. By Theorem 2.3.7, pr' is an algebraic endomorphism of A . Since A is simple, pr' cannot be a nontrivial idempotent and then $V'' = 0$, which is the first assertion.

The second assertion is an immediate consequence of the following two lemmas. Indeed, by Lemma 4.1.7, we see that the only sub representations in V_s of G_{dR}° are $V_{s, j}$'s. Since $\varphi_v(V_{i, j})$ is a sub representation of $\varphi_v(V_i) = V_s$ for some s by Lemma 4.1.6, then $\varphi_v(V_{i, j})$ is $V_{s, t}$ for some t . \square

Lemma 4.1.6. *There exists a decomposition $H_{\mathrm{dR}}^1(A, L) = \oplus V_i$ where V_i are irreducible representations of G_{dR} such that for any i , as vector spaces, $\varphi_v(V_i) = V_j$ for some j .*

Proof. When $\mathrm{End}^\circ(A)$ is a field, the decomposition is unique and any two different V_j 's are not isomorphic. In other words, V_j 's are the only irreducible sub representations of G_{dR} . Since the vector space $\varphi_v(V_i)$ is the vector space of an irreducible sub representation, it must be V_j for some j .

Moreover, G_{dR} , as a group scheme over $\mathrm{Spec} K^{\mathrm{dR}}$, is the kernel of \mathfrak{G} . Let L be the finite extension of K^{dR} such that all the fiber functors over K^{dR} are isomorphic over L . Then the extension of $\mathrm{Gal}(\bar{L}/L)$ by $G_{\mathrm{dR}}(\bar{L})$ induced by the above sequence splits. Hence \mathfrak{G} is a Galois gerb in the sense of Langlands–Rapoport.

¹¹Although $\mathrm{ad}(\varphi_v)$ defines a σ -linear automorphism of $G_{\mathrm{dR}}(K_v^{\mathrm{dR}})$, this fact itself does not imply that $G_{\mathrm{dR}, K_v^{\mathrm{dR}}}$ has a \mathbb{Q}_p -structure since a priori we do not have the cocycle condition. However, $G_{\mathrm{dR}, \overline{K_v^{\mathrm{dR}}}}$ has a \mathbb{Q}_p -structure because $\mathcal{M}_{\mathrm{dRT}} \otimes K_v^{\mathrm{dR}}$ has a fiber functor over \mathbb{Q}_p . The \mathbb{Q}_p -fiber functor can be chosen to be the étale realization because all the de Rham–Tate cycles lie in $(H_{\mathrm{ét}}^1(A_{\bar{K}_v}, \mathbb{Q}_p))^{m, n}$ via the p -adic de Rham–étale comparison.

Now we assume that the maximal subfield of $\text{End}_{\bar{K}}^{\circ}(A)$ is Galois over \mathbb{Q} . Let $\{s_{\alpha}\}$ be a \mathbb{Q} -basis of de Rham–Tate cycles in $\text{End}(H_{\text{dR}}^1(A, K^{\text{dR}}))$. By Theorem 2.3.7, these s_{α} are algebraic cycles and we use s_{α}^B to denote their image in $\text{End}(H_{\text{B}}^1(A_{\mathbb{C}}, \mathbb{Q}))$. Since A is simple, $\text{End}_{\bar{K}}^{\circ}(A)$ is a division algebra D of index d over some field F and $\{s_{\alpha}^B\}$ is a basis of D as a \mathbb{Q} -vector space. Let $E \subset D$ be a field of degree d over F . Then E is a maximal subfield of D and $D \otimes_F E \cong M_d(E)$. Therefore,

$$D \otimes_{\mathbb{Q}} E \cong D \otimes_{\mathbb{Q}} F \otimes_F E \cong D \otimes_F E \otimes_{\mathbb{Q}} F \cong M_d(E)^{[F:\mathbb{Q}]}$$

Let $e_i \in M_d(E)$ be the projection to i -th coordinate. Let $e_i^j \in D \otimes_{\mathbb{Q}} E$ be the element whose image in $M_d(E)^{[F:\mathbb{Q}]}$ is $(0, \dots, 0, e_i, 0, \dots, 0)$, where e_i is on j -th component. Since $\sum e_i^j$ is the identity element in D , there must exist at least one e_i^j such that $\sum_{\tau \in \text{Gal}(E/\mathbb{Q})} \sigma(e_i^j)$ is nonzero, where the Galois action is with respect to the \mathbb{Q} -structure of D .

We write $e_i^j = \sum k_{\alpha} s_{\alpha}^B$, where $k_{\alpha} \in E$, and let $pr_{\tau} = \sum \tau(k_{\alpha}) s_{\alpha}$, for all $\tau \in \text{Gal}(E/\mathbb{Q})$. Since e_i^j is an idempotent, so is pr_{τ} . Let V_{τ} be the image of pr_{τ} . We may assume that L contains E and still use σ to denote the image of the Frobenius via the map $\text{Gal}(L_v/\mathbb{Q}_p) \subset \text{Gal}(L/\mathbb{Q}) \rightarrow \text{Gal}(E/\mathbb{Q})$. Then by definition, as vector spaces, $\varphi_v(V_{\tau}) = V_{\sigma\tau}$.

Now we remains to prove that $\sum V_{\tau}$ is a direct sum and $\oplus V_{\tau} = H_{\text{dR}}^1(A, \bar{K})$ as representations of G_{dR} . First, since pr_{τ} lies in the centralizer of G_{dR} , every V_{τ} is a subrepresentation. Second, since the number of irreducible representations in a decomposition of $H_{\text{dR}}^1(A, \bar{K})$ equals to $[E : \mathbb{Q}]$, it suffices to prove that $\sum V_{\tau} = H_{\text{dR}}^1(A, \bar{K})$. Since the image of $\sum pr_{\tau}$ is contained in $\sum V_{\tau}$, it suffices to prove that $\sum pr_{\tau}$ is invertible. By construction, $\sum pr_{\tau}$ lies in D (via comparison) and it is nonzero by the choice of e_i^j . Therefore, $\sum pr_{\tau}$ is invertible since D is a division algebra. \square

Lemma 4.1.7. *The G_{dR}° -representations $V_{i,j}$ and $V_{i,j'}$ are not isomorphic if $j \neq j'$.*

Proof. Let T_w be a Frobenius torus of maximal rank for some finite place $w \notin \Sigma$. We only need to show that the weights of T_w acting on V_i are all different. By 4.1.3, we may consider T_w as a maximal torus of $G_{\ell}(A)$ acting on irreducible sub representations of $(G_{\ell})_{\bar{\mathbb{Q}}_{\ell}}$ in $H_{\text{et}}^1(A_{\bar{K}}, \mathbb{Q}_{\ell}) \otimes \bar{\mathbb{Q}}_{\ell}$. [Pin98, Thm. 5.10] shows that $(G_{\ell}(A), \rho_{\ell})$ is a weak Mumford–Tate pair over $\bar{\mathbb{Q}}_{\ell}$. To show the weights on V_i are different, it suffice to show that the weights of the maximal torus of each geometrical irreducible component of $(G_{\ell}(A), \rho_{\ell})$ are different. Furthermore, it reduces to the case of an almost simple component of each irreducible component. They are still weak Mumford–Tate pairs by [Pin98, 4.1]. One checks the list of simple weak Mumford–Tate pairs in [Pin98, Table 4.2] to see that all the weights are different. \square

Remark 4.1.8. The proof and hence the conclusion of this lemma are valid just under the assumption in 4.1.1.

4.2. Proof of Theorem 1.3.

4.2.1. Since the Mumford–Tate conjecture holds for all the abelian varieties considered in Theorem 1.3¹², we focus on comparing the centralizers of G_{dR} and G_{dR}° in $\text{End}(H_{\text{dR}}^1(A, \bar{K}))$. Once we prove that the centralizers of both groups are the same, we conclude by Theorem 2.3.7, Lemma 3.2.2, and Lemma 3.2.3 as in the proof of Theorem 3.2.4. We separate the cases using Albert’s classification.

Type I. Let F be the totally real field $\text{End}_K^\circ(A)$ of degree e over \mathbb{Q} . [BGK06] shows that 3.2.1 holds when g/e is odd.

Proposition 4.2.2. *If $e = g$, then 2.1.4 holds for A .*

Proof. The \mathbb{Q} -vector space $H_B^1(A, \mathbb{Q})$ has the structure of a two dimensional F -vector space. Therefore, as a $(G_{\text{MT}})_{\mathbb{Q}}$ -representation, $H_B^1(A, \bar{\mathbb{Q}})$ decomposes into g non-isomorphic irreducible sub representations of dimension two. By 4.1.3, the $G_{\text{dR}, \bar{K}}$ -representation $H_{\text{dR}}^1(A, \bar{K})$ decomposes into g non-isomorphic irreducible sub representations V_1, \dots, V_g . By Lemma 4.1.2 and 4.2.1, we only need to show that all V_i are irreducible G_{dR}° -representations. By Proposition 4.1.5, if any G_{dR}° -representation V_i is reducible, then all V_1, \dots, V_g are reducible. In such situation, all V_i decompose into one dimensional representations and hence G_{dR}° is a torus. Then by Corollary 3.3.3, A has complex multiplication, which contradicts with our assumption. \square

Remark 4.2.3. The above proof is still valid if all (equivalently, any) V_i are of prime dimension.

Now we focus on the case when $\text{End}_K(A) = \mathbb{Z}$. We refer the reader to [Pin98] for the study of 3.2.1 in this case. In particular, 3.2.1 holds when $2g$ is not of form a^{2b+1} or $\binom{4b+2}{2b+1}$, where $a, b \in \mathbb{N} \setminus \{0\}$ and in this situation, $G_\ell(A) = \text{GSp}_{2g, \mathbb{Q}_\ell}$.

Proposition 4.2.4. *Assume that $G_\ell(A) = \text{GSp}_{2g, \mathbb{Q}_\ell}$. If A is defined over a number field K which is Galois over \mathbb{Q} of degree d prime to $g!$, then 2.1.4 holds for A .*

Proof. 3.2.1 holds when $G_\ell(A) = \text{GSp}_{2g}$. It suffices to show that $H_{\text{dR}}^1(A, \bar{K})$ is an irreducible $G_{\text{dR}, \bar{K}}^\circ$ -representation. If not, then by Proposition 4.1.5, $H_{\text{dR}}^1(A, \bar{K})$ would decompose into r sub representations of dimension $2g/r$. By Corollary 3.3.3, r cannot be $2g$ and hence $r \leq g$. Let pr^j be the projection to the j th irreducible component. By Proposition 4.1.5, we have $\varphi_v(pr^j) = pr^k$ for some k and the action of φ_v on all pr^j gives rise to an element s_v in S_r , the permutation group on r elements. On the other hand, by Proposition 3.1.11, there exists a set M of rational primes of natural density 1 such that for any $p \in M$ and any $v|p$, we have that $\varphi_v^{m_v} \in G_{\text{dR}}^\circ$. Hence $s_v^{m_v}$ is the identity in S_r for such v . By the assumption, m_v is prime to $r!$ and hence s_v is trivial in S_r . In other words, pr^j is a 1-de Rham–Tate cycle. Then by Theorem 2.3.7, pr^j is algebraic, which contradicts with that A is simple. \square

Type II and III. In this case, $\text{End}_K^\circ(A)$ is a quaternion algebra D over a totally real field F of degree e over \mathbb{Q} . [BGK06] and [BGK10] shows that if $g/(2e)$ is odd, then 3.2.1 holds.

Proposition 4.2.5. *If $g = 2e$, then 2.1.4 holds for A .*

¹²The Mumford–Tate conjecture for A_i is well-studied and we will cite the results for each case in this subsection. The reduction of the conjecture for the product of A_i to the simple case is essentially contained in [Lom15, Sec. 4] and we record a proof at the end.

Proof. The $G_{\text{dR}, \bar{K}}$ -representation $H_{\text{dR}}^1(A, \bar{K})$ decomposes into form $V_1 \oplus \cdots \oplus V_g$ where V_i is two dimensional and V_i is not isomorphic to V_j unless $\{i, j\} = \{2k - 1, 2k\}$. Then we conclude by 4.2.3. \square

Type IV. In this case, $\text{End}_K^\circ(A)$ is a division algebra D over a CM field F . Let $[D : F] = d^2$ and $[F : \mathbb{Q}] = e$. Then $ed^2 | 2g$.

Proposition 4.2.6. *If $\frac{2g}{ed}$ is a prime, then G_{dR}° and G_{dR} have the same centralizer in $\text{End}_{\bar{K}}(H_{\text{dR}}^1(A, \bar{K}))$.*

Proof. We view $H_B^1(A, \mathbb{Q})$ as a F -vector space and hence view G_{MT} as a subgroup of $\text{GL}_{2g/e}$. Since the centralizer of G_{MT} is D , then $H_B^1(A, \mathbb{Q}) \otimes_F \bar{F}$ decomposes into d representations of dimension $2g/(ed)$. Hence $H_B^1(A, \bar{\mathbb{Q}})$ as a G_{MT} -representation would decompose into de representations of dimension $2g/(de)$. Then we conclude by 4.2.3. \square

Corollary 4.2.7. *If g is a prime, 2.1.4 holds for A of type IV.*

Proof. Notice that when g is a prime, then d must be 1 and e must be 2 or $2g$. The second case is when A has complex multiplication and 2.1.4 is known. In the first case we have $\frac{2g}{ed} (= g)$ being a prime. Then 2.1.4 is a consequence of Proposition 4.2.6 and the Mumford–Tate conjecture ([Chi91, Thm. 3.1]) by Lemma 3.2.3. \square

Corollary 4.2.8. *If the dimension of A is a prime and $\text{End}_{\bar{K}}(A)$ is not \mathbb{Z} , then 2.1.4 holds.*

Proof. If $g = 2$, then A has CM or is type I with $e = g$ or is type II with $g = 2e$. If g is an odd prime, then A is type I with $e = g$ or type IV. We conclude by Proposition 4.2.2, Proposition 4.2.5, and Corollary 4.2.7. \square

Proof of Theorem 1.3. 2.1.4 is equivalent for isogenous abelian varieties and then we may assume that $A = \prod A_i^{n_i}$. By 4.2.1 and the following lemma, it suffices to show that $G_{\text{dR}}^\circ(A)$ and $G_{\text{dR}}(A)$ have the same centralizer. By Lemma 4.1.2, the agreement of the centralizers of $G_{\text{dR}}^\circ(A)$ and $G_{\text{dR}}(A)$ is equivalent to that all irreducible sub representations in $H_{\text{dR}}^1(A, \bar{K})$ of $G_{\text{dR}}(A)_{\bar{K}}$ are irreducible representations of $G_{\text{dR}}^\circ(A)_{\bar{K}}$. Let V be an irreducible representation of $G_{\text{dR}}(A)_{\bar{K}}$. Since the projection $A \rightarrow A_i$ is a de Rham–Tate cycle, then there exists A_i such that $V \subset H_{\text{dR}}^1(A_i, \bar{K})$ and V is an irreducible representation of $G_{\text{dR}}(A_i)_{\bar{K}}$. Then by 2.1.5 (1), Corollary 4.2.8, and Proposition 4.2.4, the de Rham–Tate group $G_{\text{dR}}(A_i)$ is connected. Then the surjective map $G_{\text{dR}}(A) \rightarrow G_{\text{dR}}(A_i)$ is still surjective when restricted to $G_{\text{dR}}^\circ(A)$. This implies that V is an irreducible representation of $G_{\text{dR}}^\circ(A)_{\bar{K}}$. \square

Lemma 4.2.9. *Let A be as in Theorem 1.3. Then the Mumford–Tate conjecture holds for A .*

Proof. The idea of the proof is the same as that of [Lom15, Thm. 4.7]. For the simplicity of statements, we assume that each simple factor of all abelian varieties mentioned in the proof falls into one of the three cases in the assumption of the theorem. Notice that an absolutely simple abelian variety A_i of type IV either have complex multiplication or is of case (2) with $\text{End}_{\bar{K}}^\circ(A_i)$ being an imaginary quadratic field. Therefore, by assumption, A is either $B \times C_1$ or $B \times C_2^k$ where B

has no simple factor of type IV, C_1 has complex multiplication, and C_2 is absolutely simple of case (2) type IV. By [Lom15, Prop. 2.8] and corresponding statement for the Mumford–Tate group, it suffices to show that the Mumford–Tate conjecture holds for $B \times C_i$.

We first prove that the Mumford–Tate conjecture holds for B . Let $H_\ell(B)$ be the neutral connected component of the subgroup of $G_\ell(B)$ with determinant 1. Since B does not have simple factor of type IV, the group $H_\ell(B)$ is semisimple. By assumption, the Lie algebra simple component of $H_\ell(B)_{\mathbb{Q}_\ell}$ is of type C and then by [Lom15, Thm. 4.1, Rem. 4.3], the group $H_\ell(B) = \prod H_\ell(B_i)$, where $\{B_i\}$ is a set of all non-isogeny simple factors of B . Since the Mumford–Tate conjecture holds for B_i , then the conjecture holds for B by [Lom15, Lem. 3.6].

On the other hand, $H_\ell(C_1)$ is a torus and the Lie algebra of simple factors $H_\ell(C_2)_{\mathbb{Q}_\ell}$ is of type A_{p-1} for $p \geq 3$ prime¹³. Since Lie algebra of type A_{p-1} is not isomorphic to that of type C, we apply [Lom15, Prop. 3.9, Lem. 3.6] to conclude that the Mumford–Tate conjecture holds for $B \times C_i$. \square

5. A STRENGTHENING OF THE RESULT OF BOST AND ITS APPLICATION

The main result of this section is Corollary 5.2.2 and the idea is to apply Theorem 5.1.5. This result is a strengthening of Theorem 2.3.7.

5.1. A theorem of Gasbarri. For simplicity, we only work with the classical higher dimensional Nevanlinna theory developed by Griffiths and King [GK73]. See also [Bos01, Sec. 4.3] and [Gas10, Sec. 5.24].

5.1.1. Let X be a quasi-projective variety of dimension N over some number field K , P a K -point of X , and \hat{X}_P the formal completion of X at P . Let \hat{V} be a formal subvariety of \hat{X}_P of dimension d . For every complex embedding $\sigma : K \rightarrow \mathbb{C}$, we assume that there exists an analytic map $\gamma_\sigma : \mathbb{C}^d \rightarrow X_\sigma(\mathbb{C})$ which sends 0 to P_σ and maps the germ of \mathbb{C}^d at 0 biholomorphically onto the germ V_σ^{an} .

Let $z = (z_1, \dots, z_d)$ be the coordinate of \mathbb{C} and the hermitian norm $\|z\|$ be $(|z_1|^2 + \dots + |z_d|^2)^{1/2}$. Let ω be the Kahler form on $\mathbb{C}^d - \{0\}$ defined by $dd^c \log \|z\|^2$. Then ω is the pull-back of the Fubini–Study metric on $\mathbb{P}^{d-1}(\mathbb{C})$ via $\pi : \mathbb{C}^d - \{0\} \rightarrow \mathbb{P}^{d-1}(\mathbb{C})$.

Definition 5.1.2. We define the *characteristic function* $T_{\gamma_\sigma}(r)$ as follows:

$$T_{\gamma_\sigma}(r) = \int_0^r \frac{dt}{t} \int_{B(t)} \gamma_\sigma^* \eta \wedge \omega^{d-1},$$

where $B(t)$ is the ball around 0 of radius t and η is the first Chern form of a fixed Hermitian ample line bundle on a chosen compactification of X . We always assume that η is positive, which is possible by a suitable choice of the Hermitian metric.

Definition 5.1.3. We define the *order* ρ_σ of γ_σ to be $\limsup_{r \rightarrow \infty} \frac{\log T_{\gamma_\sigma}(r)}{\log r}$. It is a standard fact that ρ_σ is independent of the choice of an Hermitian ample line bundle. When ρ_σ is finite, that γ_σ is of order ρ_σ means that for any $\epsilon > 0$, we have $T_{\gamma_\sigma}(r) < r^{\rho_\sigma + \epsilon}$ for r large enough. We denote by ρ the maximum of all ρ_σ .

¹³Notice that there does not exist a non-CM abelian surface of type IV.

5.1.4. Let \mathcal{F} be an involutive subbundle of the tangent bundle T_X of X . We only focus on the case when \hat{V} is the formal leaf of \mathcal{F} passing through P . We may spread out \mathcal{F} and X and assume that they are defined over $\mathcal{O}_K[1/n]$ for some integer n . Let M_{good} be the set of finite places v of K such that $\text{char } k_v \nmid n$ and that $\mathcal{F} \otimes k_v$ is stable under p -th power map of derivatives. Let α be the density of bad places:

$$\limsup_{x \rightarrow \infty} \left(\sum_{v|p_v \leq x, v \notin M_{good}} \frac{[L_v : \mathbb{Q}_{p_v}] \log p_v}{p_v - 1} \right) \left([L : \mathbb{Q}] \sum_{p \leq x} \frac{\log p}{p - 1} \right)^{-1}.$$

Theorem 5.1.5. *Assume that \hat{V} is a formal leaf Zariski dense in X , then*

$$1 \leq \frac{N}{N - d} \rho \alpha.$$

This is a refinement of a special case of [Gas10, Thm. 5.21]. To get the better bound here using some ideas from [Her], we need the following auxiliary lemmas. Notation as in [Bos01, Sec. 4.2.1]. As in [Her], we use h_v to denote $\log \|\cdot\|_v$, where $\|p\|_v = p^{-[K_v : \mathbb{Q}_{p_v}]}$.

Lemma 5.1.6. *For any $\epsilon > 0$, any complex embedding σ , there exists a constant C_1 independent of i, D such that*

$$h_\sigma(\phi_D^i) \leq C_1(i + D) - \frac{i}{\rho_\sigma + \epsilon} \log \frac{i}{D}.$$

In particular,

$$\frac{1}{[K : \mathbb{Q}]} \sum_{\sigma} h_\sigma(\phi_D^i) \leq C_1(i + D) - \frac{i}{\rho + \epsilon} \log \frac{i}{D}.$$

Proof. This is [Gas10, Thm. 5.19 and Prop. 5.26]. We sketch a more direct proof¹⁴ for the special case here using the same idea originally due to Bost. See also [Her, Lem. 6.8].

By [Bos01, Cor. 4.16], there exists a constant B_1 only depend on d such that

$$h_\sigma(\phi_D^i) \leq -i \log r + DT_{\gamma_\sigma}(r) + B_1 i.$$

By the definition of ρ_σ , there exists a constant $M > 0$ such that for all $r > M$, we have $T_{\gamma_\sigma}(r) < r^{\rho_\sigma + \epsilon}$. On the other hand, as in the proof of [Gas10, Thm. 4.15], $-i \log r + Dr^{\rho_\sigma + \epsilon}$, as a function of r , reaches its minimum in $r_0 = (\frac{i}{(\rho_\sigma + \epsilon)D})^{1/(\rho_\sigma + \epsilon)}$. Therefore, once i/D is large enough so that $r_0 > M$, we have

$$h_\sigma(\phi_D^i) \leq -i \log r_0 + Dr_0^{\rho_\sigma + \epsilon} + B_1 i \leq -\frac{i}{\rho_\sigma + \epsilon} \log \frac{i}{D} + B_2 i,$$

for some constant B_2 . In the case when i/D is not large enough, we notice that there exists a constant B_3 such that (see for example [Bos01, Prop. 4.12])

$$h_\sigma(\phi_D^i) \leq B_3(i + D).$$

Since $\frac{i}{\rho_\sigma + \epsilon} \log \frac{i}{D} \leq B_4 i$, we have

$$h_\sigma(\phi_D^i) \leq (B_3 + B_4)(i + D) - \frac{i}{\rho_\sigma + \epsilon} \log \frac{i}{D}.$$

¹⁴We use the definition of the order as in [Bos01] rather than as in [Gas10]. Gasbarri gave a proof showing that two definitions are the same, but in this paper, we only need to work with the definition in [Bos01].

We can take C_1 to be $\max\{B_2, B_3 + B_4\}$. \square

Lemma 5.1.7. *For any $\epsilon > 0$, there exists a constant C_2 such that*

$$\frac{1}{[K : \mathbb{Q}]} \sum_{\text{all places}} h_v(\phi_D^i) \leq (\alpha + \epsilon)i \log i + C_2(i + D).$$

Proof. This is [Her, Prop. 3.6]. \square

Proof of Theorem 5.1.5. We follow [Her, Sec. 6.6]. By the slope inequality and the arithmetic Hilbert–Samuel theorem ([Bos01, Sec. 4.1, 4.2]), we have (see [Her, Sec. 6.3, Eqn. (6.10)] for reference¹⁵)

$$-C_3 D^{N+1} \leq \sum_{i=0}^{\infty} rk(E_D^i/E_D^{i+1})(C_4(i + D) + h(\phi_D^i)).$$

By Lemma 5.1.6 and Lemma 5.1.7, we have

$$-C_3 D^{N+1} \leq \sum_{i=0}^{\infty} rk(E_D^i/E_D^{i+1})(C_5(i + D) + (\alpha + \epsilon - \frac{1}{\rho + \epsilon})i \log i + \frac{i}{\rho + \epsilon} \log D).$$

Let $S_D(\delta)$ be $\sum_{i \leq D^\delta} rk(E_D^i/E_D^{i+1})(-C_5(i + D) + (-\alpha - \epsilon + \frac{1}{\rho + \epsilon})i \log i - \frac{i}{\rho + \epsilon} \log D)$ and $S'_D(\delta)$ be $\sum_{i > D^\delta} rk(E_D^i/E_D^{i+1})(-C_5(i + D) + (-\alpha - \epsilon + \frac{1}{\rho + \epsilon})i \log i - \frac{i}{\rho + \epsilon} \log D)$. By [Bos01, Lem. 4.7 (1)], $rk(E_D^0/E_D^{i+1}) < (i + 1)^d$. Hence (see [Her, Lem. 6.14]) if $\delta \geq 1$, then

$$|S_D(\delta)| \leq C_6 D^\delta \log D \sum_{i \leq D^\delta} rk(E_D^i/E_D^{i+1}) \leq C_7 D^{(d+1)\delta} \log D.$$

On the other hand, if $\frac{1}{1 - (\rho + \epsilon)(\alpha + \epsilon)} < \delta < N$, [Her, Lem. 6.15] shows that for D large enough,

$$S'_D(\delta) \geq C_8 D^{N+\delta} \log D.$$

If there exists a δ such that $1 \leq \frac{1}{1 - (\rho + \epsilon)(\alpha + \epsilon)} < \delta < N/d$, then

$$S'_D(\delta) + S_D(\delta) \geq C_9 D^{N+\delta} \log D$$

for D large enough, which contradicts with the fact that $S'_D(\delta) + S_D(\delta) \leq C_3 D^{N+1}$. In other words, $N/d \leq \frac{1}{1 - (\rho + \epsilon)(\alpha + \epsilon)}$. As ϵ is arbitrary, we obtain the desired result by rearranging the inequality. \square

5.2. A strengthening of Theorem 2.3.7. If $s \in \text{End}(H_{\text{dR}}^1(A, \bar{K}))$ is a β -cycles for $\beta > 0$, then $\varphi_v(s) = s$ for infinitely many v and then $s \in \text{Fil}^0(\text{End}(H_{\text{dR}}^1(A, \bar{K})))$ by Lemma 2.1.3. In other words, s maps $\text{Fil}^1(H_{\text{dR}}^1(A, \bar{K}))$ to itself. Since

$$H_{\text{dR}}^1(A, \bar{K})/\text{Fil}^1(H_{\text{dR}}^1(A, \bar{K})) \cong \text{Lie } A_{\bar{K}}^\vee,$$

the cycle s then induces an endomorphism \bar{s} of $\text{Lie } A_{\bar{K}}^\vee$.

Theorem 5.2.1. *Assume that $A_{\bar{K}}$ is simple. If $s \in \text{End}(H_{\text{dR}}^1(A, \bar{K}))$ is a β -de Rham–Tate cycle for some $\beta > \frac{3}{4}$, then \bar{s} is the image of some element in $\text{End}_{\bar{K}}^\circ(A^\vee)$.*

Before proving the theorem, we use it to prove a strengthening of Theorem 2.3.7.

¹⁵Although Herblot focuses on the case when $d = 1$, this inequality holds in general as all the results by Bost cited here hold in general.

Corollary 5.2.2. *Assume that $A_{\bar{K}}$ is simple. If $s \in \text{End}(H_{\text{dR}}^1(A, \bar{K}))$ is a β -de Rham–Tate cycle for some $\beta > \frac{3}{4}$, then s is algebraic.*

Proof. By Theorem 5.2.1, it suffices to show that if s is fixed by infinitely many φ_v and \bar{s} is algebraic, then s is algebraic. Since the restriction to $\text{End}_L^\circ(A)$ of the map

$$\text{Fil}^0 \text{End}(H_{\text{dR}}^1(A, L)) \rightarrow \text{End}(\text{Lie } A_L^\vee), \quad s \rightarrow \bar{s}$$

is the natural identification $\text{End}_L^\circ(A) \cong \text{End}_L^\circ(A^\vee)$, we obtain an algebraic cycle $t \in \text{End}_K^\circ(A)$ such that $\bar{t} = \bar{s}$. Then for infinitely many v , we have $\varphi_v(s - t) = s - t$ and $s - t \in \text{Fil}^1(\text{End}(H_{\text{dR}}^1(A, \bar{K})))$. By Lemma 2.1.3, $s - t = 0$ and hence s is algebraic. \square

Remark 5.2.3. The only place that we will use the assumption that $A_{\bar{K}}$ is simple is to use the first assertion of Lemma 5.2.5 to show that if s is not algebraic, then the Zariski closure of the g -dimensional formal subvariety that we will construct using s is of dimension $2g$. In general, the Zariski closure is of dimension at least $g + 1$ and the same argument as below shows that any β -cycle with $\beta > 1 - \frac{1}{2(g+1)}$ is algebraic.

5.2.4. The proof of this theorem will occupy the rest of this subsection. Since the definition of β -cycle is independent of choice of definition field and the property of being a β -cycle is preserved under isogeny, we may assume that A is principally polarized and that s is defined over K . Let X be $A^\vee \times A^\vee$ and e be its identity. The main idea is to apply Theorem 5.1.5 to the following formal subvariety $\hat{V} \subset \hat{X}_{/e}$. Consider the sub Lie algebra

$$H = \{(a, \bar{s}(a)) | a \in \text{Lie}(A^\vee)\} \subset \text{Lie}(A^\vee \times A^\vee).$$

This sub Lie algebra induces an involutive subbundle \mathcal{H} of the tangent bundle of $A^\vee \times A^\vee$ via translation. The formal subvariety \hat{V} is defined to be the formal leaf passing through e . A finite place v of K is called bad if $\mathcal{H} \otimes k_v$ is not stable under p -th power map of derivatives.

Lemma 5.2.5. *If \bar{s} is not algebraic, then the formal subvariety \hat{V} is Zariski dense in X . The A -density of bad primes is at most $1 - \beta$.*

Proof. The Zariski closure G of \hat{V} must be an algebraic subgroup of $A^\vee \times A^\vee$. The simplicity of A implies that the only algebraic subgroup of $A^\vee \times A^\vee$ with dimension larger than g must be $A^\vee \times A^\vee$. Hence if \bar{s} is not algebraic, we have $\dim G > g$ and hence $G = A^\vee \times A^\vee$.

By [Mum08, p. 138], given $v \notin \Sigma$, the p -th power map on $\text{Lie } E(A^\vee) \otimes k_v = H_{\text{dR}}^1(A, k_v)$ is the same as $\varphi_v \otimes k_v$. Therefore, for those v such that $\varphi_v(s) = s$, we have that the Lie subalgebra $\{(a, s(a)) | a \in (\text{Lie } E(A^\vee))\} \otimes k_v$ of $\text{Lie}(E(A^\vee) \times E(A^\vee)) \otimes k_v$ is closed under the p -th power map. Then $H = \{(a, \bar{s}(a)) | a \in \text{Lie}(A^\vee)\} \otimes k_v$ and $\mathcal{H} \otimes k_v$ are closed under the p -th power map. Therefore, the density of bad primes is at most one minus the density of primes satisfying $\varphi_v(s) = s$. \square

5.2.6. Let $\sigma : K \rightarrow \mathbb{C}$ be an archimedean place of K . We define γ_σ to be the composition

$$\gamma_\sigma : \mathbb{C}^g \xrightarrow{(id, \bar{s})} \mathbb{C}^g \times \mathbb{C}^g \xrightarrow{(\exp, \exp)} X_\sigma,$$

where \exp the uniformization of $\mathbb{C}^g = \text{Lie } A_\sigma^\vee \rightarrow A_\sigma^\vee$. We choose an ample Hermitian line bundle \mathcal{L} on A^\vee such that the pull back of its first Chern form via \exp is $iC_0 \sum_{k=1}^g dz_k \wedge d\bar{z}_k$ where $C_0 > 0$ is some constant. More explicitly, we may choose \mathcal{L} to be the theta line bundle with a translate-invariant metric. See for example [dJ08, Sec. 2].

To compute the order of γ_σ , we fix the ample Hermitian line bundle on X to be $pr_1^* \mathcal{L} \otimes pr_2^* \mathcal{L}$. Then

$$\gamma_\sigma^* \eta = C_0 \left(i \sum_{k=1}^g dz_k \wedge d\bar{z}_k + s^* \left(i \sum_{k=1}^g dz_k \wedge d\bar{z}_k \right) \right).$$

Thus $\gamma_\sigma^* \eta$ has all coefficients of $dz_i \wedge d\bar{z}_j$ being constant functions on \mathbb{C}^g .

Lemma 5.2.7. *The order ρ_σ of γ_σ is at most 2. In other words, $\rho \leq 2$.*

Proof. Up to a positive constant,

$$\omega = i \frac{\|z\|^2 \sum_{k=1}^g dz_k \wedge d\bar{z}_k - \sum_{k,l=1}^g \bar{z}_k z_l dz_k \wedge d\bar{z}_l}{\|z\|^4}.$$

Since all the absolute value of the coefficients of $dz_k \wedge d\bar{z}_l$ in ω are bounded by $2\|z\|^{-2}$ and those in $\gamma_\sigma^* \eta$ are constant functions, the volume form $\gamma_\sigma^* \eta \wedge \omega^{g-1}$ has the absolute value of the coefficient of $\wedge_{k=1}^g (dz_k \wedge d\bar{z}_k)$ to be bounded by $C_1 \|z\|^{-2(g-1)}$ for some constant C_1 . Hence

$$\begin{aligned} T_{\gamma_\sigma}(r) &= \int_0^r \frac{dt}{t} \int_{B(t)} \gamma_\sigma^* \eta \wedge \omega^{g-1} \\ &\leq \int_0^r \frac{dt}{t} \int_{B(t)} C_1 \|z\|^{-2(g-1)} (i^g) \wedge_{k=1}^g (dz_k \wedge d\bar{z}_k) \\ &= \int_0^r \frac{dt}{t} \int_0^t C_2 R^{-2(g-1)} \text{vol}(S(R)) dR \\ &= \int_0^r \frac{dt}{t} \int_0^t C_3 R dR = C_4 r^2, \end{aligned}$$

where $S(R)$ is the sphere of radius R . We conclude by the definition of orders. \square

Remark 5.2.8. By a more careful argument, one can see that ρ_σ is 2.

Proof of Corollary 5.2.2. If \bar{s} is not algebraic, then we apply Theorem 5.1.5 with $N = 2g$ and $d = g$. We have

$$1 \leq 2\rho\alpha \leq 2 \cdot 2 \cdot (1 - \beta),$$

which contradicts with $\beta > \frac{3}{4}$. \square

5.3. Abelian surfaces. We see from the discussion in section 4 that the only case left for 2.1.4 for abelian surfaces is when $\text{End}_{\bar{K}}(A) = \mathbb{Z}$ and K is of even degree over \mathbb{Q} . We discuss in this section the case when K is a quadratic extension of \mathbb{Q} and remark that one can deduce similar results when $[K : \mathbb{Q}]$ is $2n$ for some odd integer n by incorporating arguments as in the proof of Proposition 4.2.4.

Assume that A does not satisfy 2.1.4. Then by Proposition 4.1.5, we have $H_{\text{dR}}^1(A, \bar{K}) = V_1 \oplus V_2$ as a G_{dR}° -representation, where V_1 and V_2 are irreducible representations of dimension 2.

Let $\underline{\beta}$ be the inferior density of good primes of pr_1

$$\liminf_{x \rightarrow \infty} \left(\sum_{v|p_v \leq x, \varphi_v(pr_1)=pr_1} \frac{[L_v : \mathbb{Q}_{p_v}] \log p_v}{p_v - 1} \right) \left([L : \mathbb{Q}] \sum_{p \leq x} \frac{\log p}{p - 1} \right)^{-1}$$

and $\overline{\beta}$ be the supreme density

$$\limsup_{x \rightarrow \infty} \left(\sum_{v|p_v \leq x, \varphi_v(pr_1)=pr_1} \frac{[L_v : \mathbb{Q}_{p_v}] \log p_v}{p_v - 1} \right) \left([L : \mathbb{Q}] \sum_{p \leq x} \frac{\log p}{p - 1} \right)^{-1}.$$

By Theorem 3.1.5, for a density one set of split primes v of K , we have $\varphi_v \in G_{\text{dR}}^\circ(K_v)$ and then $\varphi_v(pr_i) = pr_i$ for $i = 1, 2$. In other words, we have $\frac{1}{2} \leq \underline{\beta} \leq \overline{\beta}$.

Theorem 5.3.1. *If A does not satisfy 2.1.4, then $\underline{\beta} \leq \frac{3}{4} \leq \overline{\beta}$. In particular, if the natural density of good primes of pr_1 exists, then the density must be $\frac{3}{4}$.*

Proof. By definition, pr_1 is a $\underline{\beta}$ -de Rham–Tate cycle. If $\underline{\beta} > \frac{3}{4}$, then by Corollary 5.2.2, we have pr_1 is algebraic. As $\text{End}_{\bar{K}}(A) = \mathbb{Z}$, this is a contradiction. Therefore, $\underline{\beta} \leq \frac{3}{4}$.

Let $\theta \in \bar{K}$ be an element such that $\sigma(\theta) = -\theta$, where σ is the nontrivial element in $\text{Gal}(K/\mathbb{Q})$. We consider $\theta pr_1 - \theta pr_2 \in \text{End}(H_{\text{dR}}^1(A, \bar{K}))$. By Proposition 4.1.5, if $\varphi_v(pr_1) \neq pr_1$, then $\varphi_v(pr_1) = pr_2$ and $\varphi_v(pr_2) = pr_1$. By the σ -linearity of φ_v , when v is inert, we have that if $\varphi_v(pr_1) \neq pr_1$, and then $\varphi_v(\theta pr_1 - \theta pr_2) = \theta pr_1 - \theta pr_2$. For v split, we have $\varphi_v(pr_i) = pr_i$ and hence $\varphi_v(\theta pr_1 - \theta pr_2) = \theta pr_1 - \theta pr_2$. Then by definition, $\theta pr_1 - \theta pr_2$ is a $\frac{3}{2} - \overline{\beta}$ -de Rham–Tate cycle. By Corollary 5.2.2, if $\overline{\beta} < \frac{3}{4}$, then $\theta pr_1 - \theta pr_2$ is algebraic, which contradicts with that $\text{End}_{\bar{K}}(A) = \mathbb{Z}$. \square

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